Quaternion Cheat Sheet and Problems
Quaternion Arithmetic

A quaternion can be represented by a vector of length four: $\dot{q} = (q_0, q_x, q_y, q_z)$. We can think of these q_i as the coefficients of a polynomial in three imaginary variables i, j, k, which is $q_0 + iq_x + jq_y + kq_z$. Quaternion arithmetic is determined by the behavior of these three imaginary variables:

$$i^2 = -1 \;\; j^2 = -1 \;\; k^2 = -1 \;\; ijk = -1$$

We can derive lots of other relationships from these, for instance, $ij = k$, $jki = -1$, or $ji = -k$. Notice that multiplication of the variables is *non-commutative*: $ij \neq ji$.

We multiply quaternions just like the polynomials they are, so that

$$\dot{q}\dot{p} = (q_0 + iq_x + jq_y + kq_z)(p_0 + ip_x + jp_y + kp_z) = q_0p_0 + iq_op_x + \ldots - q_xp_x + kq_xp_y \ldots$$

Notice $\dot{q}\dot{p} \neq \dot{p}\dot{q}$.

We can also write quaternion multiplication using matrices:

$$\dot{q}\dot{p} = \begin{bmatrix} q_0 & -q_x & -q_y & -q_z \\ q_x & q_0 & -q_z & q_y \\ q_y & q_z & q_0 & -q_x \\ q_z & -q_y & q_x & q_0 \end{bmatrix} \dot{p} = Q\dot{p}$$

and

$$\dot{p}\dot{q} = \begin{bmatrix} q_0 & -q_x & -q_y & -q_z \\ q_x & q_0 & q_z & -q_y \\ q_y & -q_z & q_0 & q_x \\ q_z & -q_y & -q_x & q_0 \end{bmatrix} \dot{p} = \bar{Q}\dot{p}$$

Notice the two matrices are different since quaternion multiplication is not commutative.

The dot-product (inner product) of two quaternions is their usual vector dot-product: $\dot{p} \cdot \dot{q} = p_0q_0 + p_xq_x + p_yq_y + p_zq_z$.

The conjugate of a quaternion, analogous to the conjugate of a complex number, is $\dot{q}^* = q_0 - iq_x - jq_y - kq_z$. Notice that $\dot{q}\dot{q}^* = q_0^2 + q_x^2 + q_y^2 + q_z^2 = \dot{q} \cdot \dot{q}$ is the squared length of $\dot{q}$, as a vector. A unit quaternion has squared length one.

Unit quaternions have three degrees of freedom. There is a two-to-one correspondence between unit quaternions and 3D rotations around an axis through the origin. A rotation of angle θ around the axis (a_x, a_y, a_z) corresponds to the quaternion $\cos\frac{\theta}{2} + \sin\frac{\theta}{2}(ia_x + ja_y + ka_z)$.

Notice that $-\dot{q} = (-q_0, -q_x, -q_y, -q_z)$ corresponds to the same rotation as $\dot{q}$; the inverse rotation is the conjugate, $\dot{q}^*$.

To apply a rotation, represented as a quaternion, to a vector v, we represent v as a *purely imaginary* quatenion $\dot{v} = iv_x + jv_y + kv_z$. Then we compute the rotated vector as

$$\dot{q}\dot{v}\dot{q}^*$$

We compose two rotations by multiplying the two quaternions. So $\dot{q}\dot{p}$ is the rotation we get by first doing $\dot{p}$, then $\dot{q}$.

1 Problems

1. Work through the process of rotating a vector $v = (x, y, z)$ by π around the z-axis in three dimensional space by writing the rotation as a quaternion q, and computing $dotq\dot{v}\dot{q}^*$ using either quaternion multiplication or the matrices above. Do a sanity check on your result!

2. For a unit quaternion $\dot{q}$, we have

$$(\dot{q}\dot{v}) \cdot (\dot{q}\dot{w}) = \dot{v} \cdot \dot{w}$$

   Use this identity to verify that $(\dot{p}\dot{q}) \cdot \dot{r} = \dot{p} \cdot (\dot{r}\dot{q}^*)$