ECS222a Graduate Algorithms
Homework 4

You are encouraged to talk to other people about these problems, but please **write up the solutions by yourself**. Always explain the answer in **your own words**; do not copy text from the book, other books, Web sites, your friends' homework, your friend's homework from last year, etc. If you use other books, Web sites, journal papers, etc. to get a solution, cite the reference and explain the solution in your own words, so that we can tell that you understand the material you are using. Always explain your solution as you would to someone who does not understand it, for instance to a beginning graduate student or an advanced undergraduate.

Please type your homework. If you know LaTeX, use that. If not, you may ype your answers in any word processing system and write in mathematical notation by hand as necessary. Include pictures if appropriate; you can draw in pictures by hand or include them in the file.

1. Markov processes with an infinite number of states arise in queuing theory. For instance, imagine we have a queue (of jobs, of messages arriving at a node of the internet, whatever...). At each time step, a new item arrives in the queue with probability p, and an item is removed from the queue with probability q. No more than one item arrives or is removed from the queue in each time step. We model the queue as a Markov process in which the state s_i is the state in which there are i elements in the queue. So there are an infinite number of states $s_0, s_1, s_2 \ldots$.
   a) For a state s_i, $i \geq 1$, draw the directed edges going into and out of s_i, and the states to which they go. Remember there may be an edge which both goes into and out of s_i. Write in the transition probability on each directed edge, as a function of p and q.
   b) Assume $p < q$. Let
   $$r = \frac{p(1-q)}{q(1-p)}$$
   Verify arithmetically that the probability distribution in which the probability of s_i is $(1-r)r^i$ is a stationary distribution for this Markov process.
   c) Is there a stationary distribution if $p > q$?

2. INDEPENDANT SET
   Input: An unweighted graph G and an integer k.
   Output: Yes if G contains a subset $U \subseteq V$ of vertices, with $|U| \leq k$, such that no two vertices in U share an edge.

   a) Show that INDEPENDANT SET is NP-complete by reduction from CLIQUE (described in Section 34.5).

   b) When G is a tree (not necessarily binary), is this problem still NP-complete? If so, prove it; if not, give a polynomial-time algorithm to find a maximum-cardinality independant set.

3. a) Formulate the optimization version of INDEPENDANT SET as a ZERO-ONE INTEGER PROGRAM (see the "problems to think about" for the lecture on Mar. 2).

   b) Ignoring the constraint that the solution must be integer, this problem becomes an LP. Write down the dual of this LP.

   c) Now constrain the solution to the dual problem to be an integer, producing another special case of INTEGER PROGRAMMING (or IP). In terms of the graph G, this IP is asking for a subset of some elements of G, which is optimal in some sense. What is the subset and how is it optimal?

   d) Prove that this IP is also NP-complete.

4. BIN-PACKING
   Input: An integer T and a list of integers $X = (x_1, x_2, \ldots x_n) \in [0, T]$.
   Output: A partition of X into a minimum number of sublists, such that each sublist sums to no more than T.

   PARTITION
   Input: A list of integers $X = (x_1, x_2, \ldots x_n)$.
   Output: YES if there is a partition of X into two lists which sum to the same value.

   PARTITION and the decision version of BIN-PACKING are NP-complete. Show that there is no polynomial time approximation algorithm for BIN-PACKING with approximation ratio less than 3/2 (unless P=NP).