ECS20 Lecture Notes on

Final Review

March 11, 2020

Review material

1. Lecture notes and related handouts
2. Homework problems sets 1–8 and solutions
3. Quizzes 1–7 and solutions
4. Midterms and solutions
5. Final review exercises and solutions

1. Set theory

   - $A = B$ iff $A \subseteq B$ and $B \subseteq A$.
   - Set operations/algebra
     $A \cup B,\ A \cap B,\ A^c,\ A \setminus B,\ A \oplus B$
   - Finite set, cardinality, and inlusion-exclusion principles
   - Power set
     $n(P(S)) = 2^{n(S)}$
   - Partition

2. Relations

- Relation $R \subseteq A \times B$
- Inverse relation $R^{-1} = \{(b,a) \mid (a,b) \in R\}$.
- Representations
- Compositions
- Types of relations:
  reflective, symmetric, antisymmetric, transitive
- Eequivalence relation and partition

3. Functions

- Composition
- One-to-one, onto, and invertible
- Frequently used functions:
  permutation, floor, ceiling, $k \pmod{m}$, exponential, logarithm
- Sequences and summations
- Recursively defined functions

4. Logic

- Proposition $\in$ {T, F}

- Compound propositions and truth tables
  negation, conjunction, disjunction, exclusive disjunction
  $\neg p,\ p \wedge q,\ p \vee q,\ p \oplus q$

- Implication, bicondition and truth tables
  $p \rightarrow q$, $p \leftrightarrow q$

- Logical equivalence ($p \leftrightarrow q$ is a tautology)

- Propositional functions $P(x)$

- Universal and existential quantifications
  $\forall x\, P(x)$, $\quad \exists x\, P(x)$

5. Proof techniques
    - Direct proof
    - Proof by contraposition
    - Proof by contradiction
    - Constructive proof
    - Proof by counterexample
    - Proof by mathematical induction

6. Integer and integer algorithms

- Divisibility $a \mid b$
- The division algorithm $a = b \cdot q + r$
- Fundamental theorem of arithmetic (prime factorization)
- $\gcd(a, b)$
- Algorithms for computing $\gcd(a, b)$

  (1) Prime factorization based algorithm

  (2) Euclidean algorithm
- Modular arithmetic $a \pmod{m}$

  Congruence relation $a \equiv b \pmod{m}$

7. Counting

- Basic rules: the sum rule and the product rule
- Inclusion-exclusion rule
  $n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$
- Mathematical functions for counting:
  - factorial
  - binomial coefficient function,
  - the binomial theorem for $(x+y)^n$,
  - Pascal's identity/triangle
- r-Permutation $P(n,r)$
- r-Combination $C(n,r)$
- The pigeonhole principle $\left\lceil \frac{N}{k} \right\rceil = m$

8. Recursion

- Counting via recursion

- Solving first-order linear recurrence relations of the form
  $a_n = c_1 a_{n-1} + c_0$

- Solving 2nd-order homogeneous linear recurrence relations with constant coefficients
  $a_n = c_1 a_{n-1} + c_2 a_{n-2}$

- Solving non-homogeneous linear recurrence relations of the form
  $a_n = c_1 a_{n-1} + f(n)$
  $a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n)$
  where $f(n)$ is some special type of function that we can start with an educated guess for a particular solution.

9. Discrete probability

- Experiment, sample space, event
- Probability of an event $p(E)$
- Probability of combinations of events
  $p(\overline{(E)}) = 1 - p(E)$
  $p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2)$
- Conditional probability $p(E|F) = \dfrac{p(E \cap F)}{p(F)}$
- Independency $p(E \cap F) = p(E) \cdot p(F)$
- Random variables*
- Distribution*
- Expectation, variance, standard derivation*
- Chebyshev's inequality*

* skip for the final

10. Graphs and trees

- Notion of a graph
  - The hand-shaking theorem
  - four ways for graph representations
  - graph isomorphism
- Special types of graphs

  K_n, C_n, W_n, Q_n, $K_{m,n}$
- Connectivity
  - counting the number of different paths using A^k
  - Euler path/cycle – edge
  - Hamilton path/cycle – vertex

10. Graphs and trees

- Planar graphs
  - Planar representation
  - Euler's formula: $v - e + r = 2$
  - Every planar graph is 4-coloable
- Tree
  - Equivalent statements of a tree
  - Rooted tree
  - m-ary tree
  - Binary search tree