Structure-Preserving Model Reduction

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Abstract. A general framework for structure-preserving model reduction by Krylov subspace projection methods is developed. The goal is to preserve any substructures of importance in the matrices L, G, C, B that define the model prescribed by transfer function $H(s) = L^*(G + sC)^{-1}B$. Many existing structure-preserving model-order reduction methods for linear and second-order dynamical systems can be derived under this general framework.

1 Introduction

Krylov subspace projection methods are increasingly popular in model reduction owing to their numerical efficiency for very large systems, such as those arising from structural dynamics, control systems, circuit simulations, computational electromagnetics and microelectromechanical systems. Recent survey articles [1,2,7] provide in depth review of the subject and comprehensive references. Roughly speaking, these methods project the original system onto a smaller subspace to arrive at a (much) smaller system having properties, among others, that many leading terms (called *moments*) of the associated (matrix-valued) transfer functions expanded at given points for the original and reduced systems match.

Consider the matrix-valued transfer function of the form

$$H(s) = L^*(G + sC)^{-1}B, \tag{1.1}$$

which describes an associated multi-input multi-output (MIMO) time-invariant system to be studied. Here $G, C \in \mathbb{C}^{N\times N}$, $B \in \mathbb{C}^{N\times m}$, $L \in \mathbb{C}^{N\times p}$. In today's applications of interests, such as VLSI circuit designs and structural dynamics, N can be up to millions [1,2,7]. Computations of $H(s)$ usually have to be done through some kind of reduction on L, G, C and B. Let $X, Y \in \mathbb{C}^{N\times n}$ such that Y^*GX is nonsingular (and thus $\mathrm{rank}(X) = \mathrm{rank}(Y) = n$). We may reduce the transfer function $H(s)$ to

$$H_{\mathrm{R}}(s) = L_{\mathrm{R}}^*(G_{\mathrm{R}} + sC_{\mathrm{R}})^{-1}B_{\mathrm{R}}, \tag{1.2}$$

where

$$L_{\mathrm{R}} = X^*L,\ G_{\mathrm{R}} = Y^*GX,\ C_{\mathrm{R}} = Y^*CX,\ B_{\mathrm{R}} = Y^*B. \tag{1.3}$$

There are various techniques to pick X and Y to perform reductions. Among them Krylov subspace-based model-reduction is getting much of the attention. The idea is

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to pick X and Y as the bases of properly defined Krylov subspaces so that $H(s) = H_{\rm R}(s) + \mathcal{O}(s^{\ell+1})$. In this paper, we show, in addition, X and Y can be chosen specially enough to preserve meaningful physical substructures.

2 Framework for Structure-Preserving Model Reduction

Suppose the matrices L, G, C, B in the transfer function (1.1) have some natural partitioning that is derived from, e.g., the physical layout of a VLSI circuit or a structural dynamical system:

$$G = \begin{array}{c} \\ N_1' \\ N_2' \end{array} \begin{array}{c} \begin{array}{cc} N_1 & N_2 \end{array} \\ \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \end{array}, \quad C = \begin{array}{c} \\ N_1' \\ N_2' \end{array} \begin{array}{c} \begin{array}{cc} N_1 & N_2 \end{array} \\ \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix} \end{array}, \tag{2.1}$$

$$L = \begin{array}{c} \\ N_1 \\ N_2 \end{array} \begin{array}{c} \begin{array}{c} p \end{array} \\ \begin{pmatrix} L_1 \\ L_2 \end{pmatrix} \end{array}, \quad B = \begin{array}{c} \\ N_1' \\ N_2' \end{array} \begin{array}{c} \begin{array}{c} m \end{array} \\ \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \end{array}, \tag{2.2}$$

where $N_1' + N_2' = N_1 + N_2 = N$. We wish that the reduced system would inherit the same structure; that is, $L_{\rm R}$, $G_{\rm R}$, $C_{\rm R}$ and $B_{\rm R}$ could be partitioned so that

$$G_{\rm R} = \begin{array}{c} \\ n_1' \\ n_2' \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{pmatrix} G_{\rm R11} & G_{\rm R12} \\ G_{\rm R21} & G_{\rm R22} \end{pmatrix} \end{array}, \; C_{\rm R} = \begin{array}{c} \\ n_1' \\ n_2' \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{pmatrix} C_{\rm R11} & C_{\rm R12} \\ C_{\rm R21} & C_{\rm R22} \end{pmatrix} \end{array}, \tag{2.3}$$

$$L_{\rm R} = \begin{array}{c} \\ n_1 \\ n_2 \end{array} \begin{array}{c} \begin{array}{c} p \end{array} \\ \begin{pmatrix} L_{\rm R1} \\ L_{\rm R2} \end{pmatrix} \end{array}, \; B_{\rm R} = \begin{array}{c} \\ n_1' \\ n_2' \end{array} \begin{array}{c} \begin{array}{c} m \end{array} \\ \begin{pmatrix} B_{\rm R1} \\ B_{\rm R2} \end{pmatrix} \end{array}, \tag{2.4}$$

with each sub-block a direct reduction from the corresponding sub-block in the original system, e.g., $G_{\rm R11}$ from G_{11}, where $n_1 + n_2 = n_1' + n_2'$. In the formulation (1.3) for the reduced system, this can be accomplished by picking

$$X = \begin{array}{c} \\ N_1 \\ N_2 \end{array} \begin{array}{c} \begin{array}{cc} n_1 & n_2 \end{array} \\ \begin{pmatrix} X_1 & 0 \\ 0 & X_2 \end{pmatrix} \end{array}, \quad Y = \begin{array}{c} \\ N_1' \\ N_2' \end{array} \begin{array}{c} \begin{array}{cc} n_1' & n_2' \end{array} \\ \begin{pmatrix} Y_1 & 0 \\ 0 & Y_2 \end{pmatrix} \end{array}, \tag{2.5}$$

such that $\mathrm{rank}(X_j) = n_j$, $\mathrm{rank}(Y_i) = n_i'$. Then the submatrices of the coefficient matrices $L_{\rm R}$, $G_{\rm R}$, $C_{\rm R}$ and $B_{\rm R}$ of the reduced system are given by

$$L_{{\rm R}j} = X_j^* L_j,\; G_{{\rm R}ij} = Y_i^* G_{ij} X_j,\; C_{{\rm R}ij} = Y_i^* C_{ij} X_j,\; B_{{\rm R}i} = Y_i^* B_i. \tag{2.6}$$

A reduction as in (2.3) – (2.6) is conceivably useful for the system matrices with meaningful substructures. For example, for the time-domain modified nodal analysis (MNA) circuit equations targeted by PRIMA [12] and SyMPVL [6], system matrices have the following natural partitioning (adopting the formulation in [6])

$$G = \begin{pmatrix} G_{11} & G_{12} \\ G_{12}^* & 0 \end{pmatrix}, \quad C = \begin{pmatrix} C_{11} & 0 \\ 0 & -C_{22} \end{pmatrix}, \quad G_{11}^* = G_{11}, \quad C_{ii}^* = C_{ii}, \quad L = B, \tag{2.7}$$

where all sub-matrices have their own physical interpretations. As proved in [12], reduction (1.3) (and thus (2.6) included) with $Y = X$ preserves passivity of the system (2.7).

Remark 1. This substructural preserving model reduction technique (2.3) – (2.6) was inspired by Su and Craig [17] concerning a second-order system which can always be equivalently turned into a linear system (see (4.2) in the next section) with a natural partitioning just as in (2.1) and (2.2).

Define the *kth Krylov subspace* generated by $A \in \mathbb{C}^{N\times N}$ on $Z \in \mathbb{C}^{N\times \ell}$ as

$$\mathcal{K}_k(A, Z) \stackrel{\text{def}}{=} \mathsf{span}(Z, AZ, \ldots, A^{k-1}Z),$$

where $\mathsf{span}(\cdots)$ is the subspace spanned by the columns of argument matrices.

Theorem 1. *Assume that G and $G_{\rm R}$ are nonsingular (and thus the total number of columns in all X_i and that in all Y_i must be the same).*

1. *If $\mathcal{K}_k(G^{-1}C, G^{-1}B) \subseteq \mathsf{span}(X)$ and $Y = X$, then $H(s) = H_{\rm R}(s) + \mathcal{O}(s^k)$.*
2. *If G and C are Hermitian, and if $\mathcal{K}_k\left(G^{-1}C, G^{-1}(B\ \ L)\right) \subseteq \mathsf{span}(X)$ and $Y = X$, then $H(s) = H_{\rm R}(s) + \mathcal{O}(s^{2k})$.*
3. *If $\mathcal{K}_k(G^{-1}C, G^{-1}B) \subseteq \mathsf{span}(X)$ and $\mathcal{K}_r(G^{-*}C^*, G^{-*}L) \subseteq \mathsf{span}(Y)$, then $H(s) = H_{\rm R}(s) + \mathcal{O}(s^{k+r})$.*

Remark 2. Theorem 1 in its generality is due to [8]. It is an extension of similar theorems in [19] for $C = I$ and $G^{-1}B = b$ (vector). For a simpler and cleaner proof based on the projector language, the reader is referred to [10].

Now that we have shown the substructures in (2.1) and (2.2) can be preserved via (2.3) – (2.6). But this is for approximating $H(s)$ around $s = 0$ only and in practice approximations to $H(s)$ around other selected points $s_0 \neq 0$ may be sought, too. Can a shift be incorporated without destroying the existing substructures? The answer is *yes*. Let s_0 be a shift and write

$$s = s_0 + (s - s_0), \tag{2.8}$$

and then

$$G + sC = G + s_0C + (s - s_0)C \stackrel{\text{def}}{=} G(s_0) + \widetilde{s}C. \tag{2.9}$$

Upon substitutions (i.e., renaming)

$$G(s_0) \to G, \quad \widetilde{s} \to s,$$

the problem of approximating $H(s)$ around $s = s_0$ becomes equivalently to approximate the substituted $H(s)$ around $s = 0$. Observe that any reduction on $G(s_0)$ and C by $Y^*G(s_0)X$ and Y^*CX can be done through reducing G and C directly as in (1.3) because

$$G_{\rm R}(s_0) \stackrel{\text{def}}{=} Y^*G(s_0)X = Y^*GX + s_0Y^*CX = G_{\rm R} + s_0C_{\rm R}. \tag{2.10}$$

This is a significant observation because it says that even for approximating $H(s)$ near a different point $s_0 \neq 0$, reduction can still be done directly to the original matrices L, G, C, and B, regardless of the shift (2.8).

As a straightforward application of Theorem 1, we have the following theorem.

Theorem 2. *Let integers $k, r \geq 0$, and let $G(s_0)$ be defined as in* (2.9)*. Assume that $G(s_0)$ and $G_{\rm R}(s_0)$ are nonsingular.*

1. *If $\mathcal{K}_k(G(s_0)^{-1}C, G(s_0)^{-1}B) \subseteq \mathsf{span}(X)$ and $Y = X$, then $H(s) = H_{\rm R}(s) + \mathcal{O}((s-s_0)^k)$.*
2. *For real s_0, if G and C are Hermitian, and if $\mathcal{K}_k\left(G^{-1}(s_0)C, G(s_0)^{-1}(B\ \ L)\right) \subseteq \mathsf{span}(X)$ and $Y = X$, then $H(s) = H_{\rm R}(s) + \mathcal{O}((s-s_0)^{2k})$.*
3. *If $\mathcal{K}_k(G(s_0)^{-1}C, G(s_0)^{-1}B) \subseteq \mathsf{span}(X)$ and $\mathcal{K}_r(G(s_0)^{-*}C^*, G(s_0)^{-*}L) \subseteq \mathsf{span}(Y)$, then $H(s) = H_{\rm R}(s) + \mathcal{O}((s-s_0)^{k+r})$.*

A sample Arnoldi-type implementation to realize Item 1 of this theorem is given below, where **strAMR** stands for *structural preserving Arnoldi-type model reduction*. Sample implementations to Items 2 and 3 can be given in a similar way.

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strAMR – Sample Implementation:
Given L, G, C, B as in (2.1) and (2.2), and expansion point s_0.

1. $\widehat{G} = G + s_0 C$; solve $\widehat{G}\widehat{Q} = B$ for $\widehat{Q}$;
2. $Q_1 = \mathtt{orth}(\widehat{Q})$: an orthonormal basis matrix for $\mathsf{span}(\widehat{G}^{-1}B)$;
3. Arnoldi process computes $\widetilde{X}$:

   For $j = 1$ to $k-1$ do
   Solve $\widehat{G}\widehat{Q} = CQ_j$ for $\widehat{Q}$;
   For $i = 1$ to j do
   $\widehat{Q} = \widehat{Q} - Q_i(Q_i^*\widehat{Q})$;
   EndFor
   $Q_{j+1} = \mathtt{orth}(\widehat{Q})$;
   EndFor

   Partition $\widetilde{X} = (Q_1\ Q_2\ \cdots\ Q_k)$ as $\widetilde{X} = \begin{matrix} N_1 \\ N_2 \end{matrix} \begin{pmatrix} \widetilde{X}_1 \\ \widetilde{X}_2 \end{pmatrix}$;

4. $X_1 = \mathtt{orth}(\widetilde{X}_1)$; $X_2 = \mathtt{orth}(\widetilde{X}_2)$; $Y_i = X_i$;
5. Compute nonzero blocks of $L_{\rm R}$, $G_{\rm R}$, $C_{\rm R}$, and $B_{\rm R}$, as in (2.6);
6. Evaluate the reduced $H_{\rm R}(s)$ as needed.

Remark 3. The invariance property (2.10) of the reduction on L, G, C, and B regardless of the shift (2.8) makes it possible to match moments at multiple points by one reduction. This is done by enforcing $\mathsf{span}(X)$ and/or $\mathsf{span}(Y)$ containing more appropriate Krylov subspaces associated at multiple points. To avoid repetition, we shall omit explicitly stating it. See [8] and Ruhe [13,14].

3 Structures of Krylov Subspaces of Block Matrices

The results of this section are of general interest. The matrices here do not necessarily have anything to do with the transfer function. Consider

$$A = \begin{array}{c} \\ N_1 \\ N_2 \end{array} \begin{array}{c} \begin{array}{cc} N_1 & N_2 \end{array} \\ \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \end{array}, \quad B = \begin{array}{c} \\ N_1 \\ N_2 \end{array} \begin{array}{c} \begin{array}{c} m \end{array} \\ \begin{pmatrix} B_1 \\ B_2 \end{pmatrix} \end{array}, \tag{3.1}$$

where $N_1 + N_2 = N$. The following theorem describes the structures in a basis matrix of $\mathcal{K}_k(A, B)$ when one of A_{ij}'s is zero.

Theorem 3. *Let A and B be partitioned as in* (3.1)*, and let* $\mathsf{span}(\widetilde{X}) = \mathcal{K}_k(A, B)$ *be partitioned as*

$$\widetilde{X} = \begin{array}{c} \\ N_1 \\ N_2 \end{array} \begin{array}{c} \begin{array}{cc} n_1' & n_2' \end{array} \\ \begin{pmatrix} \widetilde{X}_{11} & \widetilde{X}_{12} \\ \widetilde{X}_{21} & \widetilde{X}_{22} \end{pmatrix} \end{array} \equiv \begin{array}{c} \\ N_1 \\ N_2 \end{array} \begin{array}{c} \begin{array}{c} n_1'+n_2' \end{array} \\ \begin{pmatrix} \widetilde{X}_1 \\ \widetilde{X}_2 \end{pmatrix} \end{array}$$

such that $\mathsf{span}\begin{pmatrix} \widetilde{X}_{11} \\ \widetilde{X}_{21} \end{pmatrix} = \mathcal{K}_{k-1}(A, B)$*, and let $\alpha \neq 0$ be a scalar which may be different at different occurrences. Then*

1. *If $A_{11} = 0$, then* $\mathsf{span}(\widetilde{X}_1) = \mathsf{span}(B_1 \;\; A_{12}\widetilde{X}_{21}) \subseteq \mathsf{span}(B_1 \;\; A_{12}\widetilde{X}_2)$*. If in addition $A_{12} = \alpha I$ (and thus $N_1 = N_2$),* $\mathsf{span}(\widetilde{X}_1) = \mathsf{span}(B_1 \;\; \widetilde{X}_{21}) \subseteq \mathsf{span}(B_1 \;\; \widetilde{X}_2)$.
2. *If $A_{12} = 0$, then* $\mathsf{span}(\widetilde{X}_1) = \mathcal{K}_k(A_{11}, B_1)$.
3. *If $A_{21} = 0$, then* $\mathsf{span}(\widetilde{X}_2) = \mathcal{K}_k(A_{22}, B_2)$.
4. *If $A_{22} = 0$, then* $\mathsf{span}(\widetilde{X}_2) = \mathsf{span}(B_2 \;\; A_{21}\widetilde{X}_{11}) \subseteq \mathsf{span}(B_2 \;\; A_{21}\widetilde{X}_1)$*. If in addition $A_{21} = \alpha I$ (and thus $N_1 = N_2$),* $\mathsf{span}(\widetilde{X}_2) = \mathsf{span}(B_2 \;\; \widetilde{X}_{11}) \subseteq \mathsf{span}(B_2 \;\; \widetilde{X}_1)$.

Proof: All claims are consequences of the following observation:

$$\text{if} \quad A^i B = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}, \quad \text{then} \quad A^{i+1} B = \begin{pmatrix} A_{11}Z_1 + A_{12}Z_2 \\ A_{21}Z_1 + A_{22}Z_2 \end{pmatrix}.$$

Then combining the assumption that one of $A_{ij} = 0$ will complete the proof. □

Item 4 of Theorem 3 was implicitly stated in [3,4,17]. It gives a relation between $\mathsf{span}(\widetilde{X}_1)$ and $\mathsf{span}(\widetilde{X}_2)$; so does Item 1. It is Item 4 that led to structure-preserving dimension reductions of second-order systems. See § 4.

4 Model Reduction of Second-Order Systems

In this section, we show how to apply the theory presented in the previous sections to the structure-preserving model reduction of a second-order system. Consider the transfer function of a second-order system

$$H(s) = (V^* + sT^*)(s^2M + sD + K)^{-1}R, \tag{4.1}$$

where $M, D, K \in \mathbb{C}^{N\times N}$, $R \in \mathbb{C}^{N\times m}$, $T, V \in \mathbb{C}^{N\times p}$. Notation here is adopted from structural dynamics, where M, D, K are mass, damping, and stiffness matrices and are

usually Hermitian, but can be non-Hermitian at times. It is quite common to deal with (4.1) by a linearization technique to turn it into the form of (1.1). This is done by setting

$$C = \begin{pmatrix} D & M \\ M & 0 \end{pmatrix},\ G = \begin{pmatrix} K & 0 \\ 0 & -M \end{pmatrix},\ L = \begin{pmatrix} V \\ T \end{pmatrix},\ B = \begin{pmatrix} R \\ 0 \end{pmatrix}. \tag{4.2}$$

By now, all existing developments for the transfer function (1.1) can be applied in a straightforward way, but then reduced models likely lose the second-order characteristics, i.e., they may not be turned into the second-order transfer functions[3] and consequently the reduced models have little physical significance. To overcome this, Su and Craig [17] made an important observation about the structures of the associated Krylov subspaces that make it possible for the reduced second-order system to still have the second-order form

$$H_{\mathrm{R}}(s) = (V_{\mathrm{R}}^* + sT_{\mathrm{R}}^*)(s^2 M_{\mathrm{R}} + sD_{\mathrm{R}} + K_{\mathrm{R}})^{-1} R_{\mathrm{R}}, \tag{4.3}$$

where

$$\begin{array}{l} M_{\mathrm{R}} = Y_1^* M X_1,\ D_{\mathrm{R}} = Y_1^* D X_1,\ K_{\mathrm{R}} = Y_1^* K X_1, \\ V_{\mathrm{R}} = X_1^* V,\ T_{\mathrm{R}} = X_1^* T,\ R_{\mathrm{R}} = Y_1^* R. \end{array} \tag{4.4}$$

and $X_1, Y_1 \in \mathbb{C}^{N\times n}$ having full column rank. Together with L, G, C and B as defined by (4.2), the transfer functions $H(s)$ and $H_{\mathrm{R}}(s)$ of (4.1) and (4.3) takes the forms (1.1) and (1.2) with (1.3), and

$$X = \begin{array}{c} \\ N \\ N \end{array} \begin{array}{c} \begin{array}{cc} n & n \end{array} \\ \begin{pmatrix} X_1 & 0 \\ 0 & X_1 \end{pmatrix} \end{array} \quad \text{and} \quad Y = \begin{array}{c} \\ N \\ N \end{array} \begin{array}{c} \begin{array}{cc} n & n \end{array} \\ \begin{pmatrix} Y_1 & 0 \\ 0 & Y_1 \end{pmatrix} \end{array}. \tag{4.5}$$

Reduction as such for the second-order system falls nicely into our framework in §2. The question is how to construct X and Y, noticing the differences in X and Y between (2.5) and (4.5). This is where Theorem 3 comes in to help. A sample Arnoldi-type implementation **qAMR** is as follows. For more detail, see [10]. Another implementation includes the original one of [17].

5 Numerical Examples

The first example is taken from [16]. Here $N = 256$, The structure of G and C are as in Figure 1, $N_i' = N_i = 128\,(i = 1, 2)$, $p = m = 1$, and L and B are randomly chosen.

We compare the approximate accuracy of the "*structurally reduced*" models by **strAMR** as proposed against otherwise "*generally reduced*" ones, i.e., **strAMR** without Step 4 (and therefore $X = \widetilde{X}$). Figure 2 plots the values of the original and reduced transfer functions and the relative errors of the reduced functions, where $Y = X$ and $\mathsf{span}(X) \supset \mathcal{K}_{20}(G^{-1}C, G^{-1}B)$. It clearly shows that the structurally reduced model is more accurate in the long range of frequency.

[3] It is possible to turn a linear system of even dimension into a second-order system. Recently [11,15] and [9] propose two different ways to do that; but in both cases the coefficient matrices of the resulted second-order system cannot be related to the original system in a meaningful way.

qAMR – Sample Implementation: Computing X_1.

1. Compute $\widehat{X}$ such that $\mathcal{K}_q(G^{-1}C, G^{-1}(B \;\; L) \subseteq \mathsf{span}(\widehat{X})$, by, e.g., **strAMR**;
2. Partition $\widehat{X} = \begin{matrix} N \\ N \end{matrix} \begin{pmatrix} \widehat{X}_1 \\ \widehat{X}_2 \end{pmatrix}$;
3. Compute $X_1 = \mathtt{orth}((\widehat{X}_1 \;\; M^{-1}T))$.

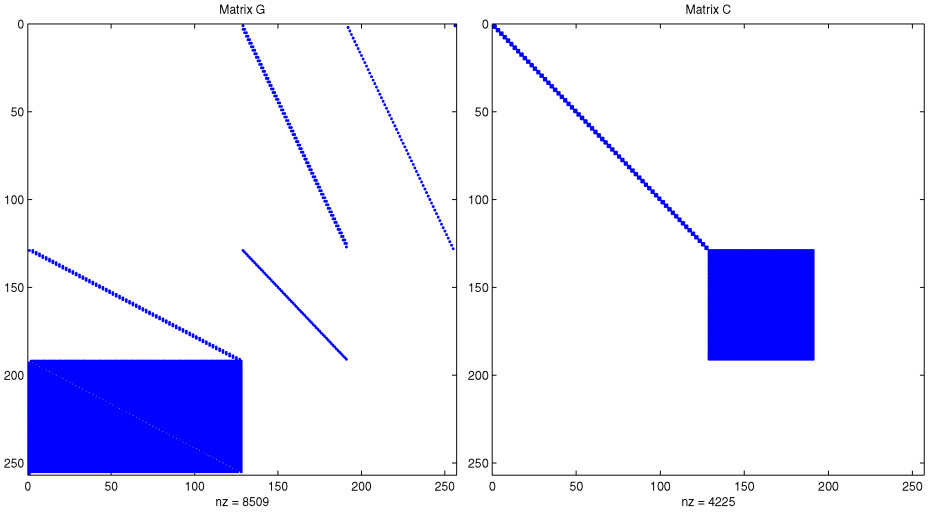

Fig. 1. Block Structure of G (left) and C (right)

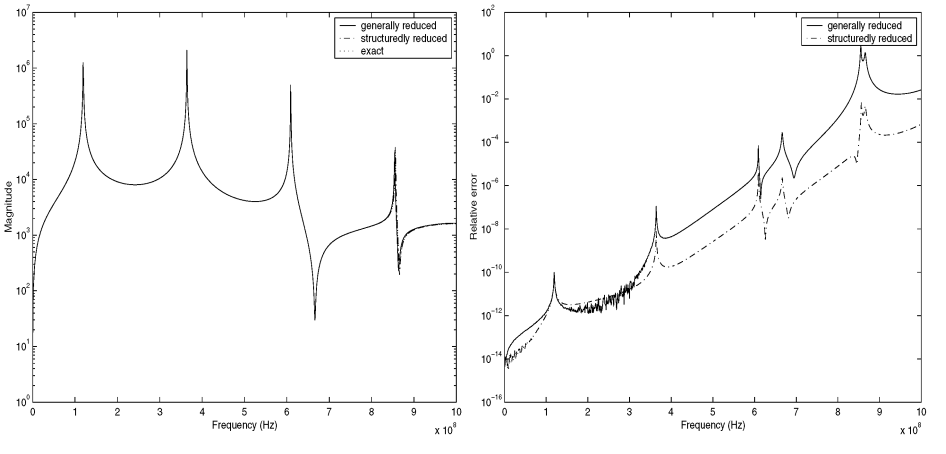

Fig. 2. Transfer functions (left) and relative errors (right)

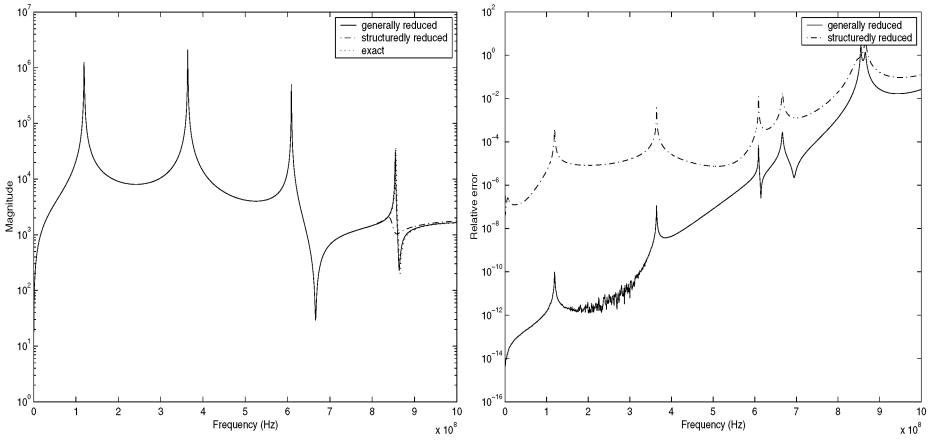

Fig. 3. Transfer functions (left) and relative errors (right); Preserving incorrect structure can lead less accurate approximations

It is natural to wonder whether incorrect structural partitioning would make any difference. Indeed it does. Supposedly we take $N_1' = N_1 = 128 + 20$ and $N_2' = N_2 = 128 - 20$. Figure 3 plots the same things as Figure 2, except with this new partition, where again $Y = X$ and $\mathsf{span}(X) \supset \mathcal{K}_{20}(G^{-1}C, G^{-1}B)$. This figure shows that improper partitioning can degrade accuracy. But more than that, for this partitioning "*structural reduction*" is less accurate than the "*general reduction*" which is quite counter-intuitive and surprising because $\mathsf{span}(X)$ with some partitioning includes $\mathsf{span}(X)$ without any partitioning, and thus a reduction with partitioning should do at least just as well as one without in terms of accuracy – further studies needed.

Next example is the second-order system from [2, §3.3]: $N = 400$, $p = m = 1$, $T = 0$, and $V = R$ randomly chosen. Figure 4 plots the values of the original and reduced transfer functions and relative errors, where "*quadratically reduced*" refers to (4.3) with (4.4) and X_1 by, e.g., **qAMR**, and "*linearly reduced*" refers to (1.2) and (1.3) through linearization (4.2) with $Y = X (= \widehat{X}$ in **qAMR** without Step 3).

6 Conclusions

A general framework for structural model reduction is established. Existing technique of Su and Craig for the second-order system can be easily realized within the framework. The idea is extensible to block partitioning with more than 2-by-2 blocks and thus makes it possible to conserve sub-structures as fine as needed for any particular system. The idea about the structures of Krylov subspaces of block matrices is not limited to 2-by-2 blocks as in Theorem 3, either and consequently the development outlined in §4 is extensible to systems of order higher than 2. Detail is in [10]. Numerical examples show the worth of the idea, as well as that incorrect identification of structures can result in poor numerical accuracy.

The work of Su and Craig [17] has spawned several recent research papers on model reduction of second-order systems and quadratic eigenvalue problems, includ-

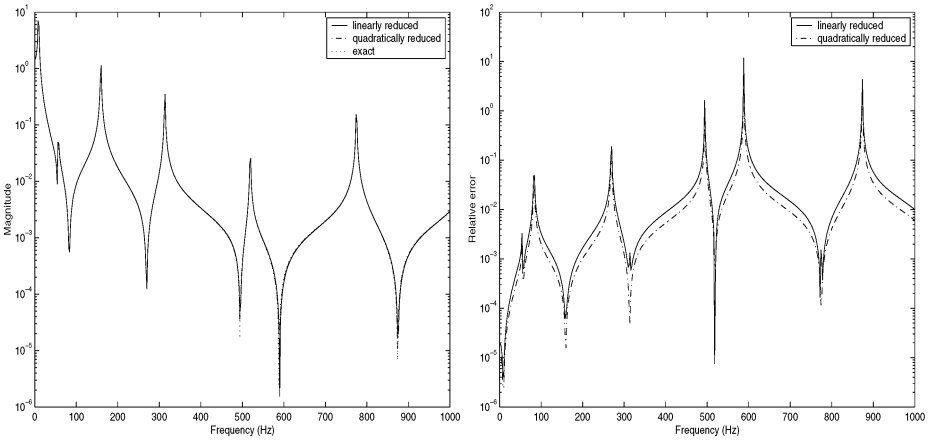

Fig. 4. Transfer functions (left) and relative errors (right): a second-order example

ing [3,4,5,18]. But the attempt to preserve meaningful substructures as in (2.3) – (2.6) for any general linear systems, not necessarily from linearizing a second-order system, appears to be conceived first by [10].

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