Graph Theory

Topics: notion of a graph, connectivity, complete and bipartite graphs, planar graphs and trees

Notion of a graph

1. A simple (undirected) graph $G = (V, E)$ consists of V, a nonempty set of *vertices*, and E, a set of unordered pairs of distinct elements of V called *edges*.

   Two vertices u and v in a graph G are called *adjacent* in G if $\{u, v\}$ is an edge of G. The edge $e = \{u, v\}$ is called *incident* with the vertices u and v.

2. The *degree* of a vertex v, denoted $\deg(v)$, is the number of edges incident with it.

3. The handshaking theorem: Let $G = (V, E)$ be a graph with e edges. Then

$$2e = \sum_{v \in V} \deg(v).$$

   Question: How many edges are there in a graph with 10 vertices each of degree 6?

4. A *subgraph* of a graph $G = (V, E)$ is a graph $G' = (V', E')$ where $V' \subseteq V$ and $E' \subseteq E$.

5. The *union* of two simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, denoted as $G_1 \cup G_2$, is the simple graph with vertex set $V_1 \cup V_2$ and edges $E_1 \cup E_2$.

6. Representing graphs:

   - By pictures
   - Using adjacency lists.
   - Using adjacency matrices: Suppose that a simple graph $G = (V, E)$ with n vertices $\{v_1, v_2, \ldots, v_n\}$. The adjacency matrix $A = (a_{ij})$ is the $n \times n$ zero-one matrix, and the element a_{ij} is defined as

$$a_{ij} = \begin{cases} 1 & \text{if } \{v_i, v_j\} \text{ is an edge of } G, \\ 0 & \textit{otherwise.} \end{cases}$$

     Note that A is symmetric.
   - Using incidence matrices: Suppose that a simple graph $G = (V, E)$ with n vertices $\{v_1, v_2, \ldots, v_n\}$ and m edges $e_1, e_2, \ldots, e_m$. The incidence matrix is the $n \times m$ $B = (b_{ij})$, where

$$b_{ij} = \begin{cases} 1 & \text{when edge } e_j \text{ is incident with } v_i, \\ 0 & \text{otherwise.} \end{cases}$$

7. The simple graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are *isomorphic* if there is a one-to-one and onto function f from V_1 to V_2 with the property that a and b in G_1 are adjacent if and only if $f(a)$ and $f(b)$ are adjacent in G_2, for all a and b in V_1. Such a function f is called an *isomorphism*.

   In other words, when two simple graphs are isomorphic, there is one-to-one correspondence between vertices of the two graphs that preserves the adjacency relationship.

8. Theorem: Simple graphs G_1 and G_2 are isomorphic if and only if for some orderings of their vertices, their adjacency matrices are equal.

9. It is generally difficult to determine whether two simple graphs are isomorphic. There are $n!$ possible one-to-one correspondences. However we can often show that two simple graphs are not isomorphic by showing that they do not share an invariant property that isomorphic simple graphs must both have, such as the same number of vertices, edges, and degrees.

Connectivity

10. A *path of length* n from v_0 to v_n in a graph is an alternating sequence of $n+1$ vertices and n edges beginning with v_0 and ending with v_n:

$$(v_0, e_1, v_1, e_2, v_2, \ldots, v_{n-1}, e_n, v_n),$$

where the edge e_i is incident with v_{i-1} and v_i.

When the graph is simple, we denote this path by its vertex sequence $v_0, v_1, v_2, \ldots, v_{n-1}, v_n$.

11. The path is a *circuit (or cycle)* if it begins and ends at the same vertex.

12. A graph is called *connected* if there is a path between every pair of distinct vertices of the graph.

13. An *Euler path* in G is a path containing every *edge* of G exactly once.

An *Euler circuit* in G is a circuit containing every *edge* of G exactly once.

14. Theorem: a connected graph has an Euler circuit if and only if each of its vertices has even degree.

15. Example: use Euler paths and circuits to solve the graph puzzles that ask you to draw a picture in a continuous motion without lifting a pencil so that no parts of the pictures is retraced. e.g. Mohammed's Scimitars.

16. A *Hamilton path* in G is a path that containing every *vertex* of G exactly once.

A *Hamilton circuit* in G is a circuit that containing every *vertex* of G exactly once.

17. Example: Around-the-world puzzle (traveling salesperson problem): is there a simple circuit contains every vertex exactly once?

18. Amazing fact: there are no known reasonable algorithm to decide if a graph is Hamiltonian. (Most computer scientists believe that no such algorithm exists.)

Complete and bipartite graphs

19. Complete graph K_n: a simple graph that contains exactly one edge between each pair of distinct vertices.

Examples: K_3, K_5

20. Bipartite graph: a simple graph G is called *bipartite* if its vertex set V can be partitioned into two disjoint nonempty sets V_1 and V_2 such that every edge connects a vertex in V_1 and a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2).

21. By a *complete bipartite graph*, denoted as $K_{m,n}$, we mean that each vertex of V_1 is connected to each vertex of V_2.

Examples: $K_{2,3}$ and $K_{3,3}$.

Planar graphs

22. A graph is called *planar* if it can be drawn in the plane without any edges crossing. Such a drawing is called a *planar representation* of graph.

23. A pictural representation of a planar graph splits the plane into *regions (faces)*, including an unbounded region.

    Euler's formula: let G be a connected planar simple graph with e edges and v vertices. Let r be the number of regions in a planar representation of G. Then

    $$r = e - v + 2.$$

24. If a graph is planar, so will be any graph obtained by removing an edge $\{u, v\}$ and adding a new vertex $\{w\}$ together with edges $\{u, w\}$ and $\{w, v\}$. Such an operation called *elementary subdivision.*

    Two graphs G_1 and G_2 are called *homeomporhic* if they can be obtained from the same graph by a sequence of elementary subdivisions.

25. Kuratowaski's Theorem. A graph is nonplanar if and only if it contains a subgraph homeomorphic to $K_{3,3}$ or K_5.

26. Theorem [Appel, Haken 1989]. Every planar graph is 4-coloable.

Trees

27. A *tree* is a connected graph with no simple circuits.

28. Theorem: A graph is a tree if and only if there is a unique simple path between any two of its vertices.

29. Let G be a graph with $n > 1$ vertices. Then the followings are equivalent:

    (a) G is a tree.

    (b) G is a cycle-free and has $n - 1$ edges

    (c) G is connected and has $n - 1$ edges.