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Handy Induction Template

Induction is a proof technique that's applied to objects that are somehow indexed by a natural number – one way to think of this index is as their "size". Most induction proofs will follow the same general pattern; the problems will look something like this.

Show that **some property P** holds for some **object** $f(n)$ **of size** n for all $n \geq N$.

Here is a strategy for approaching them.

Strategy

1. First, identify the parts of the problem: which part is the **object** that's growing as a function of n and which is the **property P**. This can be a bit confusing if the expression is an equality; usually the closed form or nicer looking formula is P.

2. Identify the smallest value m in the range for which you're asked to show that P holds. This will be your base case. If is says "for all $n \in \mathbb{N}$", then starting from zero or 1 is okay. If N is explicitly given, it might mean that the statement isn't even true for smaller values – in that case, pick N.

3. You're ready to write your base case (basis step)! Compute $f(m)$ and demonstrate that it has the property P. This is usually pretty easy.[1]

4. Now that you demonstrated that it holds for the base case m, you know that it holds for *at least one value in the world*. This means you're ready to state your **inductive hypothesis** without *being a total liar*. It will look something like this:

   Assume that the property P holds for $f(k)$ for some k.

   If you say that it holds for all k at this step then you are being a horrible liar. Stop that. You only know it works for at least one k.

   This is a good time to decompose any definitions embedded inside $f(n)$ or P to prepare yourself for what you're dealing with. If P is something like "rational", "square" or "divisible by 3", it is helpful to write out explicitly what those mean. Similarly, if $f(n)$ is the n^{th} Fibonacci number, for example, you have extra knowledge that you can use. If you're not quite comfortable with summation notation (those Σ's) or factorials, now is a good time to expand them out. These will make the next step easier.

5. Next, write down $f(k+1)$. Relax. Stare at it for a bit. Your goal right now is to try to find $f(n)$ inside of $f(k+1)$, so that you can apply the inductive hypothesis to make the problem easier! It should be really helpful or you wouldn't be doing induction.

   Stating the **inductive conclusion** at this step will often tempt you into thinking you're done. If you want to write it out and look at it, it helps to use the handy phrase *"want to show"* to remind yourself that you don't *know* it yet. The shorthand "wts" is totally legit and shows up in math papers.

6. Do you see $f(k)$ hidden inside $f(k+1)$? Great! Crank the inductive hypothesis. When you perform this step, it is a good idea to explicitly point out that you're using the inductive hypothesis; your reader may have forgotten about it, especially if you've been doing algebra already. They don't have the longest attention spans.

7. Clean up any remaining details and algebra and you should be at your inductive conclusion. Feel free to write something like *"Thus, by the principle of induction, we have shown that P holds for $f(n)$ for all $n \geq N$."*

[1] Someone asked in class about situations where you need two base cases. Usually, those happen when you find that you need to apply the inductive hypothesis twice in your inductive step; for example, once for $f(k)$ and once for $f(k-1)$. A good strategy is to always do one, and then when you get to the inductive step, go back and do another one if it turns out to be necessary.

Example

Show that the n^{th} partial sum of a geometric series $a + ar + ar^2 + \ldots + ar^{n-1}$ is given by $a\dfrac{1-r^n}{1-r}$.

1. The **growing object** is $f(n) = a + ar + ar^2 + \ldots + ar^{n-1}$. The **property** is that it's equal to $a\frac{1-r^n}{1-r}$. Now we know what's going on!

2. This problem doesn't explictly say what range n should be in, but it looks like we want to make n at least one for $f(n)$ to have any terms at all. For the base case, we show that $f(1) = ar^0 = a$ does indeed equal

   $$a\frac{\cancel{(1-r^1)}}{\cancel{1-r}} = a.$$

   Onwards!

3. Assume that

   $$a + ar + ar^2 + \ldots + ar^{k-1} = a\frac{1-r^k}{1-r}$$

   for some k. No definitions to unpack here.

4. Consider the series

   $$f(k+1) = a + ar + ar^2 + \ldots + ar^{k-1} + ar^k.$$

   Notice that we're not saying anything about what it equals or what we want it to equal yet. We're just *considering.* Also, it's nice to give these objects a name sometimes.

5. Hey wait, this looks familiar. Algebra time!

   $$\begin{aligned} f(k+1) &= a + ar + ar^2 + \ldots + ar^{k-1} + ar^k \\ &= (a + ar + ar^2 + \ldots + ar^{k-1}) + ar^k \\ &= a\frac{1-r^k}{1-r} + ar^k \text{ by the inductive hypothesis} \\ &= a\left(\frac{1-r^k}{1-r} + r^k\right) \\ &= a\left(\frac{(1-r^k) + r^k(1-r)}{1-r}\right) \\ &= a\left(\frac{(1\cancel{-r^k})\cancel{+r^k} - r^{k+1}}{1-r}\right) \\ &= a\frac{(1-r^{k+1})}{1-r}. \end{aligned}$$

   The principle of induction says we're done!