1. Introduction
2. Combinations with Repetitions
   2.1. For M kinds of objects, the number of combinations of r such objects is C($r + M - 1$, r) $=\binom{r+M-1}{r}$ $= C(r + M - 1, M - 1) = \binom{r+M-1}{M-1} = \frac{(r+M-1)!}{r!(M-1)!}$.
   2.1.1. Such combinations with repetitions are called "multisets."
   2.2. Example: How many 3-combinations, with repetition allowed, can be selected from {1, 2, 3, 4}.
   2.2.1. List: [{1, 1, 1}; {1, 1, 2}; {1, 1, 3}; {1, 1, 4}; {1, 2, 2}; {1, 2, 3}; {1, 2, 4}; {1, 3, 3}; {1, 3, 4}; {1, 4, 4}; {2, 2, 2}; {2, 2, 3}; {2, 2, 4}; {2, 3, 3}; {2, 3, 4}; {2, 4, 4}; {3, 3, 3}; {3, 3, 4}; {3, 4, 4}; {4, 4, 4}] for 20 multisets.
   2.2.2. Think of representing the combinations using three |'s to separate the different categories, and three x's indicating the choice. So {1, 1, 2} would be "xx|x||", {2, 3, 4} would be "|x|x|x", and {3, 3, 4} would be "||xx|x". So we must place three x's in any of six positions. Note that there are $M - 1$ dividers, so the total places is $r + M - 1$. The combinations of choosing 3 from six, $\binom{6}{3} = \frac{6!}{3!(6-3)!} = \frac{6*5*4}{3*2} = 20 = C(r + M - 1, r)$.
3. Ordered and Unordered Partitions
   3.1. The number m of ordered partitions of a set S with n elements into r cells [$A_1, A_2, \ldots A_r$] where for each i, $n(A_i) = n_i$, is: $m = \frac{n!}{n_1!n_2!\ldots n_r!}$
   3.2. Example: Ten students needed to be placed in 3 rooms to take a test. The first room holds 2 students, the second holds 3 students, and obviously the last room holds 5 students. How many ways can the students be placed in the three rooms?
   3.2.1. There are $\binom{10}{2}$ ways to fill the first room, which leaves $\binom{8}{3}$ ways to fill the second room, and $\binom{5}{5} = 1$ ways to fill the last room. So, by the product rule we have $m = \binom{10}{2}\binom{8}{3} = \frac{10!}{2!8!} * \frac{8!}{3!5!} = \frac{10!}{2!3!5!} = \frac{10*9*8*7*6}{2*3*2} = 10 * 9 * 4 * 7 = 2520$.
4. Inclusion-Exclusion Principle Revisited (skipped)
5. Pigeonhole Principle Revisited (skipped)
6. Recurrence Relations
   6.1. A "recurrence relation" for a sequence $a_0, a_1, a_2, \ldots$ is an equation that relates each term a_n to one or more of its predecessors in the sequence, namely, $a_{n-1}, a_{n-2}, \ldots, a_{n-i}$ where i is an integer with $n - i \geq 0$.
   6.1.1. For Fibonacci, $i = 2$, since $a_n = a_{n-1} + a_{n-2}$
   6.1.2. For compound interest, $i = 1$, since $a_n = (1 + \text{interest})a_{n-1}$
   6.2. There may be many sequences that satisfy a recurrence relation, e.g. both 2, 3, 5, 8… and 7, 8, 15, 23, 38… satisfy the Fibonacci recurrence relation.
   6.3. The "initial conditions" for such a recurrence relation specify the values of $a_0, a_1, a_2, \ldots, a_{i-1}$, if i is a fixed integer, or $a_0, a_1, a_2, \ldots, a_m$, where m is an integer with $m \geq 0$, if i depends on n.
   6.3.1. A given set of initial conditions for a recurrence relation may specify a unique sequence.
   6.4. A formula for a_n in terms of n and not of the previous terms, is called a "solution" of the recurrence relation.
   6.4.1. Example: The sequence of Catalan numbers arise in a remarkable variety of different contexts of combinatorics. The solution to Catalan recurrence relation is $C_n = \frac{1}{n+1}\binom{2n}{n}$ for $n \geq 1$

   For example, C_n is the number of different ways n +1 factors can be completely parenthesized (or the number of ways of associating n applications of a binary operator). For $n = 3$, $C_n = 5$, and we have the following five different parenthesizations of four factors: *((ab)c)d, (a(bc))d, (ab)(cd), a((bc)d), a(b(cd))*.

   Show that the Catalan sequence satisfies the recurrence relation $F_n = \frac{4n-2}{n+1}F_{n-1}$ for all $n \geq 2$.

We must first determine the initial condition for the recurrence relation. $C_1 = \frac{1}{1+1}\binom{2}{1} = \frac{1}{2} * \frac{2}{1*1} = 1$, so we will let $F_1 = 1$.

Now we will use induction to prove the assertion.

Basis step: $C_2 = \frac{1}{2+1}\binom{2*2}{2} = \frac{1}{3} * \frac{4*3}{2} = 2$, and $F_2 = \frac{4*2-2}{2+1} * 1 = \frac{6}{3} = 2$, so $C_2 = F_2$ which is what we needed to show for the basis step.

Inductive hypothesis: $C_k = F_k,\ that\ is\ \frac{1}{k+1}\binom{2k}{k} = F_k$

We must show that $C_{k+1} = F_{k+1}, that\ is \frac{1}{(k+1)+1}\binom{2(k+1)}{k+1} = \frac{4(k+1)-2}{(k+1)+1}F_k$ by starting with the left side of the equation and showing that it is equal to the right side by using the inductive hypothesis.

Assertion	**Reason**
$\frac{1}{(k+1)+1}\binom{2(k+1)}{k+1}$	left side of inductive conclusion
$=\frac{1}{(k+1)+1}\frac{(2(k+1))!}{(k+1)!(k+1)!}$	by the formula for *n* choose *r*
$=\frac{1}{(k+1)+1}\frac{2(k+1)(2k+1)2k!}{(k+1)k!(k+1)k!}$	by definition of factorial
$=\frac{1}{(k+1)+1}\frac{2(k+1)(2k+1)2k!}{(k+1)(k+1)k!k!}$	by rearranging factors
$= \frac{1}{(k+1)+1}\frac{2(k+1)(2k+1)}{k+1}\frac{1}{k+1}\binom{2k}{k}$	by the formula for *n* choose *r*
$= \frac{1}{(k+1)+1}\frac{2(k+1)(2k+1)}{k+1}F_k$	by the inductive hypothesis
$= \frac{2(2k+1)}{((k+1)+1)}F_k$	by collecting terms and cancelling a k+1 term
$= \frac{4(k+1)-2}{((k+1)+1)}F_k$	by rearranging factors. QED

7. Linear Recurrence Relations with Constant Coefficients
   7.1. A recurrence relation of order k is a function of the form: $a_n = \Phi(a_{n-1}, a_{n-2}, \ldots, a_{n-k}, n)$
   7.2. A "linear k^{th} order recurrence with constant coefficients" is a recurrence relation of the form:
      $a_n = C_1a_{n-1} + C_2a_{n-2} + \ldots + C_ka_{n-k} + f(n)$ where $C_1, C_2, \ldots, C_k$ are constants with $C_k \neq 0$, and $f(n)$ is a function of n.
      7.2.1. "Linear" = There are no powers or products of the a_i's.
      7.2.2. "Constant coefficients" = The constants, $C_1, C_2, \ldots, C_k$ are constants, and not dependent on n.
      7.2.3. "Homogenous" = If $f(n) = 0$.
   7.3. We can solve for a_n, if given initial k values of $a_{n-1}, a_{n-2}, \ldots, a_{n-k}$.
8. Solving Second-order Homogenous Linear Recurrence Relations
   8.1. Theorem 6.8: Suppose the characteristic polynomial $\Delta(x) = x^2 - sx - t$ of the second-order homogenous linear recurrence relation with constant coefficients: $a_n = sa_{n-1} + ta_{n-2}$, has distinct roots r_1 and r_2. Then the general solution of the recurrence relation follows, where c_1 and c_2 are arbitrary constants: $a_n = c_1r_1^n + c_2r_2^n$ with c_1 and c_2 computed from initial conditions.
   8.2. The sequence for Example 6.10 in the book is incorrect it should be: 1, 2, 7, 20, 61, 182
   8.3. Example #1: Find the solution for $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$.
      8.3.1. This is in the form $a_n = sa_{n-1} + ta_{n-2}$ where $s = 1$ and $t = 2$.
      8.3.2. The characteristic equation is $\Delta(x) = x^2 - sx - t = x^2 - x - 2 = (x - 2)(x + 1)$, so the distinct roots are $r_1 = 2$, and $r_2 = -1$.
      8.3.3. From Theorem 6.8, the general form of the solution is $a_n = c_1r_1^n + c_2r_2^n = c_12^n + c_2(-1)^n$
      8.3.4. From the initial conditions we have
         $a_0 = 2 = c_12^0 + c_2(-1)^0 = c_1 + c_2$
         $a_1 = 7 = c_12^1 + c_2(-1)^1 = 2c_1 - c_2$
      8.3.5. Solving using $c_1 = 2 - c_2$ we have $a_1 = 7 = 2(2 - c_2) - c_2 = 4 - 2c_2 - c_2 = 4 - 3c_2$ so $3 = -3c_2$, and thus $c_2 = -1$, and $c_1 = 2 - c_2 = 2 - (-1) = 3$
      8.3.6. The solution of the recurrence relation is $a_n = c_1r_1^n + c_2r_2^n = 3*2^n - (-1)^n$

8.4. Example #2: Find the solution for $f_n = f_{n-1} + f_{n-2}$, with the initial conditions $f_0 = 0$, and $f_1 = 1$. Fibonacci

8.4.1. This is in the form $a_n = sa_{n-1} + ta_{n-2}$ where $s = 1$ and $t = 1$.

8.4.2. The characteristic equation is $\Delta(x) = x^2 - sx - t = x^2 - x - 1$, from the quadratic formula $x = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$ we have $r_1 = \frac{1+\sqrt{5}}{2}$, and $r_2 = \frac{1-\sqrt{5}}{2}$

From Theorem 6.8, the general form of the solution is $a_n = c_1 r_1^n + c_2 r_2^n = c_1(\frac{1+\sqrt{5}}{2})^n + c_2(\frac{1-\sqrt{5}}{2})^n$

8.4.3. From the initial conditions we have

$$f_0 = 0 = c_1(\frac{1+\sqrt{5}}{2})^0 + c_2(\frac{1-\sqrt{5}}{2})^0 = c_1 + c_2$$

$$f_1 = 1 = c_1(\frac{1+\sqrt{5}}{2})^1 + c_2(\frac{1-\sqrt{5}}{2})^1$$

8.4.4. Solving using $c_1 = -c_2$, we have $f_1 = 1 = -c_2\left(\frac{1+\sqrt{5}}{2}\right) + c_2(\frac{1-\sqrt{5}}{2}) = c_2(\frac{-1-\sqrt{5}+(1-\sqrt{5})}{2}) = c_2(\frac{-2\sqrt{5}}{2}) = -c_2\sqrt{5}$, so $c_2 = \frac{-1}{\sqrt{5}}$ and $c_1 = \frac{1}{\sqrt{5}}$

8.4.5. The solution for the Fibonacci recurrence is $f_n = c_1 r_1^n + c_2 r_2^n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^n - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^n$

8.5. Solutions when Roots of the Characteristic Polynomial are Equal, Theorem 6.9: Suppose the characteristic polynomial $\Delta(x) = x^2 - sx - t$ of the second-order homogenous linear recurrence relation with constant coefficients: $a_n = sa_{n-1} + ta_{n-2}$, <u>has only one root r_0</u>. Then the general solution of the recurrence relation follows, where c_1 and c_2 are arbitrary constants: $a_n = c_1 r_0^n + c_2 n r_0^n$ with c_1 and c_2 computed from initial conditions.

8.6. Example #3: Find the solution for the recurrence $a_n = 6a_{n-1} - 9a_{n-2}$ with the initial conditions $a_0 = 1$, and $a1 = 6$.

8.6.1. This is in the form $a_n = sa_{n-1} + ta_{n-2}$ where $s = 6$ and $t = 9$.

8.6.2. The characteristic equation is $\Delta(x) = x^2 - sx - t = x^2 - 6x - 9 = (x - 3)^2$, so there is only one root, $r_0 = 3$.

8.6.3. From Theorem 6.9, the general form of the solution is $a_n = c_1 r_0^n + c_2 n r_0^n = c_1 3^n + c_2 n 3^n$

8.6.4. From the initial conditions we have

$$a_0 = 1 = c_1 3^0 + c_2 * 0 * (3)^0 = c_1$$

$$a_1 = 6 = c_1 3^1 + c_2 * 1 * (3)^1 = 3c_1 + 3c_2$$

8.6.5. From a_0 we know $c_1 = 1$, so $a_1 = 6 = 3 + 3c_2$, thus $3 = 3c_2$, and $1 = c_2$

8.6.6. The solution for the recurrence is $a_n = c_1 r_0^n + c_2 n r_0^n = 1*3^n + 1*n3^n = (1 + n)3^n$

9. Solving General Homogeneous Linear Recurrence Relations (skipped)