1. Intro to Sets.
   1.1. Let $D = \{1, 3, 5, 7\}$, $E = \{3, 4, 5\}$, $F = \{2, 4, 6\}$ for this handout.
2. Sets and elements
   2.1. Definition of set: *A*n unordered, but well-defined, collection of objects called "elements" (or "members") of the set. The objects may be of any sort, e.g. integers, letters, ordered pairs, vertices of a graph, edges of a graph, or students.
   2.2. Specifying sets
      2.2.1. Enumerated list of elements inside braces, separated by commas: $D = \{1, 3, 5, 7\}$ or $E = \{3, 5, 4\}$
         2.2.1.1. Duplicated objects, and re-orderings are ignored, e.g. $\{1, 3, 3, 5\} = \{1, 5, 3\}$
      2.2.2. Stating property(s) of the elements inside braces, with properties separated by commas, e.g., $D =\{x \mid x \text{ is an odd integer}, 0 < x < 8\}$, where "|" means "such that"
   2.3. Subset = a set A is a "subset" of set B if every element of A is an element of B, and is written $A \subseteq B$.
      2.3.1. Two sets are "equal" if they both have the same elements, i.e., $A = B$ iff $A \subseteq B$ and $B \subseteq A$.
         2.3.1.1. Since $A \subseteq A$, then $A = A$.
      2.3.2. A is a "proper subset" of B if every element of A is an element of B, but A does not contain all of the elements of B. $A \subset B$ iff $A \subseteq B$ and $A \neq B$.
      2.3.3. If $A \subseteq B$ and $B \subseteq D$, then $A \subseteq D$.
   2.4. Special Set Symbols
      2.4.1. $\mathbf{Z}$ = set of all integers. $\{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$
      2.4.2. $\mathbf{N}$ = set of all natural numbers. $\{0, 1, 2, 3, \ldots\}$ or $\{x \mid x \in \mathbf{Z}, x \geq 0\}$
      2.4.3. $\mathbf{R}$ = set of all real numbers.
      2.4.4. $\mathbf{Q}$ = set of all rational numbers, i.e., quotients of integers. $\{x \mid x \in \mathbf{R}, x = p/q, p \in \mathbf{Z}, q \in \mathbf{Z}, q \neq 0\}$
      2.4.5. $\mathbf{C}$ = set of all complex numbers
      2.4.6. $\mathbf{N} \subseteq \mathbf{Z} \subseteq \mathbf{Q} \subseteq \mathbf{R} \subseteq \mathbf{C}$
   2.5. Universal Set and Empty Set
      2.5.1. $\mathbf{U}$ = Universal set. All elements in the sets in the current application are assumed to belong to a fixed large set called the universal set.
      2.5.2. $\varnothing$ = the empty set, which contains no elements. There is only one empty set, and it is subset of every other set.
      2.5.3. For any set A, $\varnothing \subseteq A \subseteq \mathbf{U}$
   2.6. Disjoint Sets = two sets that have no elements in common.
      2.6.1. Of sets D,E, and F, only sets D and F are disjoint.
3. Venn Diagrams = a pictorial representation of sets in which sets are enclosed areas in the plane. Used with quantified statements.
   3.1. The universal set is the interior of a rectangle.
   3.2. If $A \subseteq B$, then the area of A will be contained within the area of B.
   3.3. If A and B are disjoint, then the area of A will be completely separate from the area of B.
   3.4. If $A \cap B \neq \varnothing$, then the area of A will overlap part of the area of B.
   3.5. Examples: Use a Venn diagram to show the validity or invalidity of the following arguments:
      3.5.1. Premises: All human beings are mortal. Zeus is not mortal. Conclusion: Zeus is not a human being.
      3.5.2. Premises: All human beings are mortal. Felix is mortal. Conclusion: Felix is a human being.
      3.5.3. Premises: No dorm food is good. No good food is wasted. Conclusion: No dorm food is wasted.
4. Set Operations: Let $D = \{1, 3, 5, 7\}$, $E = \{3, 4, 5\}$
   4.1. Union = set of all elements that belong to A or B. $D \cup E = \{1, 3, 4, 5, 7\}$
      4.1.1. Property: $A \subseteq A \cup B$, and $B \subseteq A \cup B$.
   4.2. Intersection = set of all elements that belong to both A and B. $D \cap E = \{3, 5\}$

4.2.1. Property: $A \cap B \subseteq A$, and $A \cap B \subseteq B$

4.3. Theorem: The following are equivalent $A \subseteq B$, $A \cap B = A$, $A \cup B = B$.

4.4. Complement = A^C is the set of all elements of the universal set that are not in A. Let U = **Z**, then $E^C = \{\ldots, -1, 0, 1, 2, 6, 7, \ldots\}$

4.5. Difference = A - B (or $A \backslash B$ in the text) is the set of elements that belong to A but not B, e.g. $D - E = \{1, 7\}$, and $E - D = \{4\}$

4.6. Symmetric difference = $A \oplus B$ is those elements that belong to one set, but not the other, e.g. $D \oplus E = \{1, 4, 7\}$

4.6.1. $A \oplus B = (A \cup B) - (A \cap B) = (A - B) \cup (B - A)$

4.7. Fundamental products (skipped)

5. Algebra of Sets

5.1. Principle of Duality

5.1.1. The "dual" of an equation is obtained by replacing each occurrence of $\cup$, $\cap$, **U**, and $\varnothing$ with $\cap$, $\cup$, $\varnothing$, and **U** respectively.

5.1.2. If any equation is an identity, then its dual is also an identity.

5.2. Laws of the algebra of sets

Name	Union version	Intersection Dual version
Commutative	$A \cup B = B \cup A$	$A \cap B = B \cap A$
Associative	$(A \cup B) \cup D = A \cup (B \cup D)$	$(A \cap B) \cap D = A \cap (B \cap D)$
Distributive	$A \cup (B \cap D) = (A \cup B) \cap (A \cup D)$	$A \cap (B \cup D) = (A \cap B) \cup (A \cap D)$
Idempotent	$A \cup A = A$	$A \cap A = A$
Identity	$A \cup \varnothing = A$	$A \cap \mathbf{U} = A$
	$A \cup \mathbf{U} = \mathbf{U}$	$A \cap \varnothing = \varnothing$
Complement	$A \cup A^C = \mathbf{U}$	$A \cap A^C = \varnothing$
	$\mathbf{U}^C = \varnothing$	$\varnothing^C = \mathbf{U}$
Double Complement	$(A^C)^C = A$	
De Morgan's	$(A \cup B)^C = A^C \cap B^C$	$(A \cap B)^C = A^C \cup B^C$
Absorption	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
Set Difference	$A - B = A \cap B^C$	

5.3. Methods of Set Proofs

5.3.1. Membership tables. Similar to truth tables of propositional logic, except T = $x \in$ set expression.

5.3.1.1. Example: Prove $(A \cap B) \cup (A^C \cap B) = B$ using Method 1

A	B	$A \cap B$	A^C	$(A^C \cap B)$	$(A \cap B) \cup (A^C \cap B)$
T	T	T	F	F	T
T	F	F	F	F	F
F	T	F	T	T	T
F	F	F	T	F	F

5.3.2. Convert to a problem in propositional logic, prove, then convert back

5.3.2.1. Example: Prove the De Morgan's Law that states $(A \cap B)^C = A^C \cup B^C$.

Assertion	Reason
$(A \cap B)^C$	premise
$= \{x \mid x \in (A \cap B)^C\}$	by definition of set notation
$= \{x \mid x \notin (A \cap B)\}$	by definition of complement
$= \{x \mid \neg[x \in (A \cap B)]\}$	by definition of not an element symbol
$= \{x \mid \neg(x \in A \wedge x \in B)\}$	by definition of intersection
$= \{x \mid \neg(x \in A) \vee \neg(x \in B)\}$	by De Morgan's law of logical equivalence
$= \{x \mid (x \notin A) \vee (x \notin B)\}$	by definition of not an element symbol.
$= \{x \mid (x \in A^C) \vee (x \in B^C)\}$	by definition of complement
$= A^C \cup B^C$	by definition of set notation

5.3.3. Use set identities for a tabular proof (similar to what we did for the propositional logic examples but using set identities)

5.3.3.1. Prove: $A \cup B = (A \cap B^C) \cup (A^C \cap B) \cup (A \cap B)$ using set identities

Assertion	Reason
$(A \cap B^C) \cup (A^C \cap B) \cup (A \cap B)$	premise
$= (A \cap B^C) \cup [(A^C \cup A) \cap B]$	by distributive law
$= (A \cap B^C) \cup (\mathbf{U} \cap B)$	by complement law
$= (A \cap B^C) \cup B$	by identity law
$= (A \cup B) \cap (B^C \cup B)$	by distributive law
$= (A \cup B) \cap \boldsymbol{U}$	by complement law
$= A \cup B$	by identity law

5.3.4. Prove $[(A \cup B) \subseteq (A \cap B)] \rightarrow (A = B)$ by using an indirect proof of contraposition.

We must show that $(A \neq B) \rightarrow \neg[(A \cup B) \subseteq (A \cap B)]$

Since $A \neq B$ then $\exists x$ such that| $(x \in A$ and $x \notin B)$ or $(x \notin A$ and $x \in B)$. We can use a proof by cases

Assertion	**Reason**
1. $A \neq B$	premise
2. $\exists x \mid (x \in A \wedge x \notin B)$ or $(x \notin A \wedge x \in B)$	(1) by definition of unequal sets
3. $\exists x \mid (x \in A \wedge x \notin B)$	(2) and first case for the premise that must proved
4. $c \in A \wedge c \notin B$	(3) by existential instantiation
5. $c \in (A \cup B)$	(4) by definition of union
6. $c \notin (A \cap B)$	(4) by definition of intersection
7. $\neg[(A \cup B) \subseteq (A \cap B)]$	(5) and (6) by definition of subset since $A \cup B$ contains an element that is not in $A \cap B$.
8. $\exists x \mid (x \notin A \wedge x \in B)$	(2) and second case for the premise that must proved
9. $c \notin A \wedge c \in B$	(8) by existential instantiation
10. $c \in (A \cup B)$	(9) by definition of union
11. $c \notin (A \cap B)$	(10) by definition of intersection
12. $[(A \cup B) \subseteq (A \cap B)]$	(10) and (11) by definition of subset since $A \cup B$ contains an element that is not in $A \cap B$.
13. $(A \neq B) \rightarrow \neg[(A \cup B) \subseteq (A \cap B)]$	(7) and (12) using proof by cases
14. $[(A \cup B) \subseteq (A \cap B)] \rightarrow (A = B)$	(13) by proof by contraposition.

6. Finite Sets, Counting Principle

6.1. A set is "finite" if it contains exactly m elements, where m is a non-negative integer, otherwise it is "infinite."

6.2. n(S) denotes the number of elements in a finite set S, e.g. n(D) = 4, and n(E) = 3.

6.2.1. if A and B are finite sets and $A \cap B = \varnothing$ (disjoint sets), then $n(A \cup B) = n(A) + n(B)$

6.2.2. if A and B are finite sets (disjoint or not), then $n(A \cup B) = n(A) + n(B) - n(A \cap B)$

6.3. Venn diagrams can be used to answer questions about the number of elements in the intersections of sets.

6.3.1. Example: Given the following facts about the Winter course schedules for first year Computer Science majors answer the questions. There are 350 first year CS majors. 100 are taking ECS 20, 50 are taking ECS 30, and 200 are taking ECS 40. 75 are taking both ECS 20 and ECS 40. 10 are taking ECS 20 and ECS 30. One crazy student is taking all three courses!

6.3.1.1. How many CS students are taking only ECS 40?

6.3.1.2. How many CS students are not taking any of the three courses?

6.3.1.3. How many CS students are taking more than one of the three courses?

7. Classes of Sets, Power Sets, Partitions

7.1. A set of sets is called a "class of sets" to avoid confusion.

7.2. Power Set = P(S) = the class of all subsets of S, e.g. P(E) = [∅, {3}, {4}, {5}, {3, 4}, {3, 5}, {4, 5}, {3, 4, 5}]

7.2.1. If S is finite, then $n(P(S)) = 2^{n(S)}$, e.g. $n(P(E)) = 2^{n(E)} = 2^3 = 8$.

7.2.2. Example: Determine the factors of 330 efficiently. An inefficient method would be to test each value from 2 to 165 (330 / 2) to see if each is a divisor of 330. A more efficient way would be to determine the prime factors of 330 by testing only the primes between 2 and $\lfloor\sqrt{330}\rfloor = 18$, so 2, 3, 5, 7, 11, 13, and 17. This would result in 2, 3, 5, and 11. Let S = {2, 3, 5, 11}. The factors are then one and P(S), where we multiply each subset's elements together, and ignore the empty set. There are $n(P(S)) = 2^{n(S)} = 2^4 = 16$ factors of 330.

7.3. Partition = a partition of S is a class of non-empty subsets of S such that every element of S belongs to exactly one of the subsets. Therefore the sets in a given partition are mutually disjoint.

7.3.1. The possible partitions of E are [{3}, {4}, {5}], [{3}, {4, 5}], [{4}, {3,5}], [{5}, {3, 4}], and [{3, 4, 5}]. This is a list of all the possible ways to combine the elements of E.

7.4. Generalized Set Operations = set operations applied to any number of sets.

7.4.1. For this section, let there be a finite collection of sets A_1, A_2, … A_m, and another collection of sets that is uncounted, G.

7.4.2. $A_1 \cup A_2 \cup \ldots \cup A_m = \bigcup_{i=1}^{m} A_i = \{x \mid x \in A_i \text{ for some } A_i\}$, and $\bigcup(B \mid B \in G) = \{x \mid x \in B_i \text{ for some } B_i \in G\}$

7.4.2.1. Example: Let $H_i = \{i, 2i\}$, then $\bigcup_{i=1}^{4} H_i = \{1, 2\} \cup \{2, 4\} \cup \{3, 6\} \cup \{4, 8\} = \{1, 2, 3, 4, 6, 8\}$

7.4.3. $A_1 \cap A_2 \cap \ldots \cap A_m = \bigcap_{i=1}^{m} A_i = \{x \mid x \in A_i \text{ for every } A_i\}$, and $\bigcap(B \mid B \in G) = \{x \mid x \in B_i \text{ for every } B_i \in G\}$

7.4.3.1. Example: Let $H_i = \{i - 1, i, i + 1\}$, then $\bigcap_{i=-1}^{1} H_i = \{-2, -1, 0\} \cap \{-1, 0, 1\} \cap \{0, 1, 2\} = \{0\}$