Data, Logic, and Computing

ECS 17 (Winter 2024)

Patrice Koehl
koehl@cs.ucdavis.edu

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Discussion 7: Proofs

Exercise 1

Let n be an integer. Show that if $2n^2 + n + 9$ is odd, then n is even using an indirect proof, a proof by contradiction, and a direct proof.

This is a problem of showing a conditional $p \to q$ is true, where
$p : 2n^2 + n + 9$ is odd
$q : n$ is even

We will use three different types of proof: indirect, proof by contradiction, and direct

a) Indirect proof: we show that $\neg q \to \neg p$ is true

   Hypothesis: $\neg q$ is true, namely n is odd.

   Since n is odd, there exists an integer k such that $n = 2k + 1$. Therefore, $2n^2 + n + 9 = 2(2k + 1)^2 + (2k + 1) + 9 = 8k^2 + 10k + 12 = 2(4k^2 + 5k + 6)$

   Since $4k^2 + 5k + 6$ is integer, $2n^2 + n + 9$ is even, therefore $\neg p$ is true. Therefore $\neg q \to \neg p$ is true, and $p \to q$ is true.

b) Proof by contradiction: we suppose $p \to q$ is false

   Hypothesis: $p \to q$ is false, i.e. p is true AND $\neg q$ is true, namely $2n^2 + n + 9$ is odd and n is odd.

   Since n is odd, there exists an integer k such that $n = 2k + 1$. Therefore, $2n^2 + n + 9 = 2(2k + 1)^2 + (2k + 1) + 9 = 8k^2 + 10k + 12 = 2(4k^2 + 5k + 6)$

   Since $4k^2 + 5k + 6$ is integer, $2n^2 + n + 9$ is even. But we have supposed that $2n^2 + n + 9$ is odd. We have reached a contradiction. Therefore the hypothesis we made is false, therefore $p \to q$ is true.

c) Direct proof: we show directly that $p \to q$ is true.

   Hypothesis: p is true, $2n^2 + n + 9$ is odd. Therefore there exists an integer k such that $2n^2 + n + 9 = 2k + 1$, i.e. $n = 2k - 2n^2 - 8 = 2(k - n^2 - 4)$. Since $k - n^2 - 4$ is an integer, we conclude that 2 divides n, therefore n is even. We have showed that q is true, therefore $p \to q$ is true

Exercise 2

Let p be a natural number. Show that $2^{\frac{1}{4}}$ is irrational.

We use a proof by contradiction: let us suppose that $2^{\frac{1}{4}}$ is a rational number. There exists two integers a and b, with $b \neq 0$ such that

$$2^{\frac{1}{4}} = \frac{a}{b} \tag{1}$$

After raising this equation to the power 2, we get:

$$\sqrt{2} = \frac{a^2}{b^2} \tag{2}$$

As a and b are integers; a^2 and b^2 are integers, with $b^2 \neq 0$. The equation above would then mean that $\sqrt{2}$ is rational; this is not true. Therefore $2^{\frac{1}{4}}$ is irrational.

Exercise 3

Let a and b be two integers. Show that if either ab or $a+b$ is odd, then either a or b is odd

This is an implication of the form $p \to q$, with:

p: ab is odd or $a+b$ is odd

q: a is odd or b is odd

where a and b are integers.

We use an indirect proof (proof by contrapositive).
Hypothesis: $\neg q$: a is even and b is even.
There exist two integers k and l such that $a = 2k$ and $b = 2l$. Then
$ab = 2k \times 2l = 4kl = 2(2kl)$ therefore there exists an integer $m(= 2kl)$ such that $ab = 2m$: ab is even.
and
$a + b = 2k + 2l = 2(k + l)$ therefore there exists an integer $n(= k + l)$ such that $a + b = 2n$: $a + b$ is even.
We have proved that ab is even and $a + b$ is even; $\neg p$ is true. Therefore $\neg q \to \neg p$ is true, and by contrapositive, $p \to q$ is true.

Exercise 4

Let a and b be two integers. Show that if $a^2(b^2 - 2b)$ is odd, then a is odd and b is odd.

This is an implication of the form $p \to q$, with:

p: $a^2(b^2 - 2b)$ is odd

q: a is odd and b is odd

where a and b are integers.
We use an indirect proof (proof by contrapositive).
Hypothesis: $\neg q$: a is even or b is even. We look at both cases:

Case 1: a is even.

There exits an integer k such that $a = 2k$. Then $a^2(b^2 - 2b) = 4k^2(b^2 - 2b) = 2[2k^2(b^2 - 2b)]$. Since $2k^2(b^2 - 2b)$ is an integer, we conclude that $a^2(b^2 - 2b)$ is even.

Case 2: b is even.

There exits an integer l such that $b = 2l$. Then $a^2(b^2 - 2b) = a^2(4l^2 - 4l) = 2[a^2(2l^2 - 2l)]$. Since $a^2(2l^2 - 2l)$ is an integer, we conclude that $a^2(b^2 - 2b)$ is even.

In both cases we have shown that $a^2(b^2 - 2b)$ is even, i.e. that $\not p$ is true. Therefore $\neg q \to \neg p$ is true, and by contrapositive, $p \to q$ is true.