

Data, Logic, and Computing

ECS 17 (Winter 2022)

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Final: solutions

Part 1: Data (*10 questions, each 3 points; total 30 points*)

- 1) *CPUs in computers are covered with a heat sink to:*
 - a) *Dissipate the heat they release*
 - b) *Make them look pretty*
 - c) *Protect them from shocks (for example if the computer falls)*
 - d) *Isolate them from the other devices to avoid interferences*

- 2) *Let A be the number with the binary representation 100 and B the number whose hexadecimal representation is 1; which of these numbers X (in hexadecimal form) satisfies $BX^2 - AX + A = 0$?*
 - a) *A*
 - b) *1*
 - c) *2*
 - d) *D*

Note that:

- a) $A = (100)_2 = 4$ (decimal)
- b) $B = \#1 = 1$ (decimal)

Therefore $BX^2 - AX + A = X^2 - 4X + 4 = (X - 2)^2$, therefore $X = 2$ when $BX^2 - AX + A = 0$, and $X = (2)_{10} = \#2$.

- 3) *Which word is encoded in the ASCII code 01100101 01100011 01110011?*
 - a) *ect*
 - b) *ess*
 - c) *ECS*

d) *ecs*

Note that:

a) $01100101 = 0110\ 0101 = \#65$: letter e

b) $01100011 = 0110\ 0011 = \#63$: letter c

c) $01110011 = 0111\ 0010 = \#73$: letter s

and the letter *P* is represented by the hexadecimal #50. Therefore the correct answer is a).

4) *A continuous-time signal is uniformly sampled over 5001 samples, for a total of 2.5 seconds. Components with which of those frequencies could be correctly detected in the sampled signal? (circle all that apply)*

a) *400 Hz*

b) *800 Hz*

c) *1200 Hz*

d) *1600 Hz*

Samples are collected every $\frac{2.5}{5000} = 0.0005$ second, i.e. at a frequency of 2000 Hz. Only components with frequencies strictly lower than 1000 Hz can be detected correctly.

5) *You take a picture with a digital camera, and you know that this picture requires 32 Mbytes of storage (without compression). Assuming that each pixel is stored on 32 bits, what is the resolution of your camera, in megapixels:*

a) *16*

b) *1*

c) *8*

d) *32 s*

Each pixel needs 32 bits, i.e. 4 bytes. Since you store the picture on 32MBytes, the resolution of the camera is 8 MBytes.

6) *Which binary number comes right after the binary number 110111?*

a) *111000*

b) *110111*

c) *111111*

d) *110112*

7) *The binary representation of the hexadecimal 95 is?*

a) *1011001*

b) *1011111*

c) *10010101*

d) 149

$$\#95 = 95 = (10010101)_2.$$

8) You want to store a movie on your computer. You know that your movie is 2 hour long. It was filmed at a rate of 25 frames per second and each frame requires 10 kilobytes of storage. The soundtrack was stored in stereo, recorded at 50KHz, with 2 bytes per point. How much space is needed to store the whole movie and its soundtrack, in megabytes (assuming that 1 megabyte = 1000 kilobytes)?

a) 2520 megabytes (=2.52 GB)

b) 3240 megabytes (=3.24 GB)

c) 1440 megabytes (=1.44GB)

d) 1800 megabytes (=1.8 GB)

images: $2 \text{ (hours)} \times 3600 \text{ (second/hour)} \times 25 \text{ (frames)} \times 10 \text{ (kilobytes)} = 1,800,000 \text{ (kilobytes)}$
 $= 1,800 \text{ megabytes.}$

sound: $2 \text{ (hours)} \times 3600 \text{ (second/hour)} \times 50 \text{ (KHz)} \times 2 \text{ (bytes)} \times 2 \text{ (stereo)} = 1,440,000$
 $\text{(kilobytes)} = 1440 \text{ megabytes)}$

Total: $1800+1440=3240$ megabytes.

9) One of these four sentences would NOT be considered as a proposition in Logic:

a) All cats are white

b) An apple is not a fruit

c) $2+4=8$

d) $X+2=8$

$X + 2 = 8$ is not a proposition as we do not what X is.

10) What is the largest number (unsigned integer) that can be stored on one bit?

a) 255

b) 256

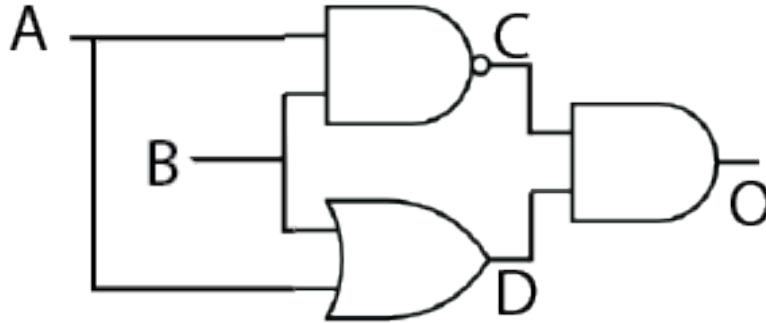
c) 128

d) 1

Part II 3 problems, each 10 points; total 30 points)

1) Complete the logic table corresponding to the logic gate shown below. Convert it into a Boolean expression (10 points)

The corresponding Boolean expression is $\overline{AB}(\overline{A} + B)$.



A	B	$C = \overline{AB}$	$D = \overline{A} + B$	$O = CD = \overline{AB}(\overline{A} + B)$
1	1	0	1	0
1	0	1	0	0
0	1	1	1	1
0	0	1	1	1

- 2) *A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet three inhabitants: Alex, John and Sally. Alex says, "If Sally is a knight, I am a knave." John says, "Alex and Sally are of the same type." Sally claims, "I like chocolate." (Note that "to be of the same type" means that they are both knights or they are both knaves.) Does Sally really like chocolate? Justify your answer.*

We check all possible "values" for Alex, John, and Sally, as well as the veracities of their statements:

Line number	Alex	John	Sally	Alex says	John says
1	Knight	Knight	Knight	F	T
2	Knight	Knight	Knave	T	F
3	Knight	Knave	Knight	F	T
4	Knight	Knave	Knave	T	F
5	Knave	Knight	Knight	T	F
6	Knave	Knight	Knave	T	T
7	Knave	Knave	Knight	T	F
8	Knave	Knave	Knave	T	T

We can eliminate:

- Line 1, as Alex would be a knight but he lies
- Line 2, as John would be a knight but he lies
- Line 3, as Alex would be a knight but he lies
- Line 5, as Alex would be a knave but he tells the truth
- Line 6, as Alex would be a knave but he tells the truth
- Line 7, as Alex would be a knave but he tells the truth

– Line 8, as Alex would be a knave but he tells the truth

The only option is Line 4; Sally is a knave, she does not like chocolate!

- 3) *Let p , q , and r be three propositions. Use a truth table or logical equivalence to indicate if the proposition $(p \wedge q) \vee r \vee (\neg q \wedge \neg r) \vee (\neg p \wedge \neg r)$ is a tautology, a contradiction, or neither*

$$\begin{aligned}(p \wedge q) \vee r \vee (\neg q \wedge \neg r) \vee (\neg p \wedge \neg r) &\Leftrightarrow (p \wedge q) \vee r \vee (\neg q \vee \neg p) \wedge \neg r \\ &\Leftrightarrow (p \wedge q) \vee r \vee (\neg(p \wedge q)) \wedge \neg r\end{aligned}$$

Let us define $A = (p \wedge q) \vee r$. Then,

$$\begin{aligned}(p \wedge q) \vee r \vee (\neg q \wedge \neg r) \vee (\neg p \wedge \neg r) &\Leftrightarrow A \vee \neg A \\ &\Leftrightarrow T\end{aligned}$$

Therefore $(p \wedge q) \vee r \vee (\neg q \wedge \neg r) \vee (\neg p \wedge \neg r)$ is a tautology.

Part III *4 problems, each 10 points; total 40 points*

- 1) *Let n be an integer. Show that n is even if and only if $n + n^2 - n^3$ is even.*

Let p be the proposition “ n is even” and q be the proposition “ $n + n^2 - n^3$ is even”. We want to show that $p \leftrightarrow q$ is true, which is logically equivalent to show that $p \rightarrow q$ and $q \rightarrow p$.

- i) Let us show $p \rightarrow q$:

Hypothesis: p is true, i.e. n is even. If n is even, then there exists an integer k such that let $n = 2k$, We get:

$$\begin{aligned}n + n^2 - n^3 &= 2k + 4k^2 - 8k^3 \\ &= 2(k + 2k^2 - 4k^3)\end{aligned}$$

As $k + 2k^2 - 4k^3$ is an integer, $n + n^2 - n^3$ is a multiple of 2: it is even. We have shown that q is true when p is true: $p \rightarrow q$ is true.

- ii) Let us show $q \rightarrow p$:

We use an indirect proof, i.e. we show that $\neg p \rightarrow \neg q$. Hypothesis: $\neg p$ is true, i.e. n is odd. If n is odd, then there exists an integer k such $n = 2k + 1$. We get:

$$\begin{aligned}n + n^2 - n^3 &= 2k + 1 + 4k^2 + 4k + 1 - 8k^3 - 12k^2 - 6k - 1 \\ &= -8k^3 - 8k^2 + 1 \\ &= 2(-4k^3 - 4k^2) + 1\end{aligned}$$

As $-4k^3 - 4k^2$ is an integer, $n + n^2 - n^3$ is odd. We have shown that $\neg q$ is true when $\neg p$ is true: $\neg q \rightarrow \neg p$ is true, and therefore $p \rightarrow q$ is true.

We conclude: “ n is even” and “ $n + n^2 - n^3$ is even” are logically equivalent.

- 1) *Let a and b be two integers. Show that $a^2 - 4b \neq 3$ (hint: you may assume true the fact that when n is an integer, if n^2 is odd, then n is odd).*

Let:

$$P: a^2 - 4b \neq 3$$

where a and b are two integers. To show that P is true, we use a proof by contradiction, namely we assume P is false:

$$a^2 - 4b = 3$$

We get:

$$a^2 = 3 + 4b = 2(1 + 2b) + 1$$

As $1 + 2b$ is an integer, a^2 is odd, and therefore a is odd (from the hint. There exists an integer k such that $a = 2k + 1$. Replacing above, we get:

$$\begin{aligned}(2k + 1)^2 - 4b &= 3 \\ 4k^2 + 4k + 1 - 4b &= 3 \\ 4k^2 + 4k - 4b &= 2\end{aligned}$$

After division by 2,

$$2k^2 + 2k - 2b = 1$$

$2k^2 + 2k - 2b = 2(k^2 + k - b)$ and since $k^2 + k - b$ is an integer, $2k^2 + 2k - 2b$ is even. However, 1 is odd: we have reached a contradiction. Therefore P is true.

- 3) *Show that:*

$$\forall n \in \mathbb{N}, \sum_{i=1}^n \frac{2}{3^i} = 1 - \frac{1}{3^n}$$

We want to Prove : $P(n)$ is true, for all $n \geq 1$, where

$$P(n): \sum_{i=1}^n \frac{2}{3^i} = 1 - \frac{1}{3^n}$$

Let us define $LHS(n) = \sum_{i=1}^n \frac{2}{3^i}$, and $RHS(n) = 1 - \frac{1}{3^n}$.

– *Basis step:* We want to prove $P(1)$ is true.

$$LHS(1) = \frac{2}{3},$$

and

$$RHS(1) = 1 - \frac{1}{3} = \frac{2}{3}.$$

Therefore, $LHS(1) = RHS(1)$: $P(1)$ is true.

- *Inductive step:* Let $P(n)$ be true for an integer $n \geq 1$, which means $LHS(n) = RHS(n)$. To prove that $P(n+1)$ is true, we prove that $LHS(n+1) = RHS(n+1)$. Let us compute $LHS(n+1)$:

$$\begin{aligned}
 LHS(n+1) &= \sum_{i=1}^{n+1} \frac{2}{3^{i+1}} \\
 &= \sum_{i=1}^n \frac{2}{3^i} + \frac{2}{3^{n+1}} \\
 &= LHS(n) + \frac{2}{3^{n+1}} \\
 &= RHS(n) + \frac{2}{3^{n+1}} \\
 &= 1 - \frac{1}{3^n} + \frac{2}{3^{n+1}} \\
 &= 1 - \frac{3}{3^{n+1}} + \frac{2}{3^{n+1}} \\
 &= 1 - \frac{1}{3^{n+1}}
 \end{aligned}$$

and

$$RHS(n+1) = 1 - \frac{1}{3^{n+1}}$$

Therefore $LHS(n+1) = RHS(n+1)$, i.e. $P(n+1)$ is true.

According to the principle of mathematical induction, we can conclude that $\sum_{i=1}^n \frac{2}{3^i} = 1 - \frac{1}{3^n}$ for all $n \geq 1$.

- 4) *Let a_k be the sequence defined by $a_k = a_{k-1} + k + 4$ for $k \geq 2$, with $a_1 = 5$. Show that*

$$\forall n \in \mathbb{N}, a_n = \frac{n(n+9)}{2}$$

We want to Prove : $P(n)$ is true, for all $n \geq 1$, where

$$P(n): a_n = \frac{n(n+9)}{2}$$

Let us define $LHS(n) = a_n$, and $RHS(n) = \frac{n(n+9)}{2}$.

- *Basis step:* $LHS(1) = a_1 = 5$,

and

$$RHS(1) = \frac{1(1+9)}{2} = 5.$$

Therefore, $LHS(1) = RHS(1)$: $P(1)$ is true.

- *Inductive step:* Let $P(n)$ be true for an integer $n \geq 1$, which means $LHS(n) = RHS(n)$. To prove that $P(n+1)$ is true, we prove that $LHS(n+1) = RHS(n+1)$. Let us compute

$LHS(n + 1)$:

$$\begin{aligned}LHS(n + 1) &= a_{n+1} \\ &= a_n + n + 1 + 4 \\ &= LHS(n) + n + 5 \\ &= RHS(n) + n + 5 \\ &= \frac{n(n + 9)}{2} + n + 5 \\ &= \frac{n(n + 9) + 2n + 10}{2} \\ &= \frac{n^2 + 11n + 10}{2} \\ &= \frac{(n + 1)(n + 10)}{2}\end{aligned}$$

and

$$RHS(n + 1) = \frac{(n + 1)(n + 10)}{2}$$

Therefore $LHS(n + 1) = RHS(n + 1)$, i.e. $P(n + 1)$ is true.

According to the principle of mathematical induction, we can conclude that $a_n = \frac{n(n+9)}{2}$ for all $n \geq 1$.

Part IV *Extra credit: 5 points*)

How many bitstrings of length $n \geq 2$ can we form that contain at least one 0 and at least one 1?

Let S be the set of bit strings of length n . There are 2^n such bitstrings. Let A be the subset of those bit strings that contain at least one 0, and let B be the subset of those bit strings that contain at least one 1. We are interested in $C = A \cap B$. To find the cardinality of C , we consider instead \overline{C} , i.e. the subset of bit strings that either do not contain one 0, or do not contain one 1: $\overline{C} = \overline{A \cap B} = \overline{A} \cup \overline{B}$.

. Based on the sum rule, $|\overline{C}| = |\overline{A}| + |\overline{B}| - |\overline{A} \cap \overline{B}|$.

- \overline{A} is the set of bit strings of length n that do not contain any 0. $|\overline{A}| = 1$.
- \overline{B} is the set of bit strings of length n that do not contain any 1: $|\overline{B}| = 1$.
- $\overline{A} \cap \overline{B}$ is the set of bitstrings that do not contain 0s and 1s: $|\overline{A} \cap \overline{B}| = 0$.

Then, $|\overline{C}| = 1 + 1 = 2$, and $|C| = 2^n - 2$.