Data, Logic, and Computing

ECS 17 (Winter 2023)

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Final: solutions

Part 1: Data *(10 questions, each 3 points; total 30 points)*

1) *1) What is the largest unsigned integer that can be stored on 2 bytes?*

   a) *256*
   b) *255*
   c) *65535*
   d) *65536*

   2 bytes = 16 bits; largest unsigned integer is $2^16 - 1 = 65535$

2) *2) Convert the binary number* $(1101101111110101)_2$ *to hexadecimal*

   a) *#DBF5*
   b) *#DCF5*
   c) *#5FBD*
   d) *#5FCD*

3) $(1110)_2 - (101)_2 =$

   a) *#B*
   b) *#8*
   c) *#A*
   d) *#9*

   Note that:

   a) $(1110)_2 = (14)_{10}$
   b) $(101)_2 = (5)_{10}$

   Therefore $(1110)_2 - (101)_2 = (9)_{10} = \#9$.

4) *4) Which of these sampling rates would be appropriate for a sound sample of maximum frequency 16 kHz (circle all that apply)?*

   a) *16000 Hz*
   b) *8000 Hz*
   c) *35000 Hz*
   d) *35 Hz*

   Samples need to be collected at a rate strictly larger that 2×16000 Hz, i.e. at a rate strictly larger that 32000 Hz.

5) *5) Assume that you have taken a square picture with a 4 megapixel digital camera. Assume that you are printing this picture out on a printer that has approximately 4000 dots per inch. How big would the picture be? (note: 1 dot = 1 pixel)*

   a) *1 inch* $\times$ *1inch*
   b) *2 inches* $\times$ *2 inches*
   c) *0.5 inch* $\times$ *0.5 inch*
   d) *4 inches* $\times$ *4 inches*

   The picture include 4,000,000 pixels; as it is square, it has 2000 pixels along both dimensions. 2000 pixels corresponds to 0.5 inch.

6) *6) Which binary number comes right after the binary number 101?*

   a) *1110*
   b) *111*
   c) *102*
   d) *110*

7) *7) Decode the name whose ASCII representation is #53 #61 #6C #6C #79*

   a) *Sally*
   b) *Sylla*
   c) *SALLY*
   d) *SYLLA*

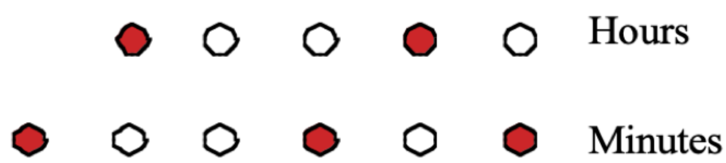
8) *8) The highest frequency note for a piano is fc=4200 Hz. Assuming that you record 1 hour of piano music with a sampling rate 3 times fc, in mono, with 16 bits resolution, what is the size of the resulting file (assuming 1MB = 1,000,000 bytes):*

   a) *0.9072 MB*
   b) *90.72 MB*
   c) *9.072 MB*
   d) *181.44 MB*

Storage: 1 (hour) × 3600 (second/hour) × 3 × 4200 (samples) × 16 (bits) × 1 (mono) = 725760000 (bits) = 90720000 bytes = 90.72 megabytes.

Total: 1800+720=2520 megabytes.

9) *9) What time is it on this digital clock (filled circle mean on)?*



a) *10:37*

b) *10:41*

c) *18:37*

d) *18:41*

10) *10) You want to store an electronic copy of a book on your computer. This book contains 500 pages; each page contains (on average) 60 lines, and each line contains 60 characters (again, on average), including space. Each character needs 2 bytes of storage. How much space do you need to store this book (assuming 1MB = 1,000,000 bytes)?*

a) *3.6 MB*

b) *36 MB*

c) *0.36 MB*

d) *360 MB*

Size = 500 (pages) × 60 (lines) × 60 (characters) × 2 (bytes). = 3600000 (bytes) = 3.6 MB.

Part II *3 problems, each 10 points; total 30 points)*

1) *For each of the five propositions in the table below, indicates on the right if they are always tautologies or not (p and q are propositions).* (10 points)

Table 1: Propositional logic

Proposition	Tautology (Yes/ No)
If 2+6=5, then 10=-9	Yes: This is $p \to q$ where p is false: therefore $p \to q$ is true
$(p \vee \neg p) \to q$	No: $p \vee \neg p$ is always true, but the implication is true only if q is true.
$(p \wedge \neg q) \to p$	Yes: $(p \wedge \neg q) \to p \equiv (\neg p \vee q \vee p \equiv T \vee q \equiv T$
if 3+3=6 then 25=16+9	Yes! This is $p \to q$ where p and q are true: therefore $p \to q$ is true
$(p \wedge \neg p) \vee (\neg p \vee p)$	Yes! $(p \wedge \neg p) \vee (\neg p \vee p) \equiv F \vee T \equiv T$

2) *A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You go to this island, as you have been told that a treasure may be buried on it. You meet two inhabitants, John, and Sally. John tells you that, "I am a knight if and only if the treasure is on the island." Sally tells you that "If John is a knight, then the treasure is not on the island." Can it be determined if the treasure is on the island? Can it be determined also whether John is a knight or knave? What about Sally ? Justify your answers. (10 points)*

John and Sally can be a knight or a knave, and the treasure is either present on the island or not.

Line number	John	Sally	Treasure on island	John says	Sally says
1	Knight	Knight	Yes	T	F
2	Knight	Knight	No	F	T
3	Knight	Knave	Yes	T	F
4	Knight	Knave	No	F	T
5	Knave	Knight	Yes	F	T
6	Knave	Knight	No	T	T
7	Knave	Knave	Yes	F	T
8	Knave	Knave	No	T	T

We can eliminate:

- Line 1, as Sally would be a knight but she lies
- Line 2, as John would be a knight but he lies
- Line 4, as John would be a knight but he lies
- Line 6, as John would be a knave but he tells the truth
- Line 7, as Sally would be a knave but she tells the truth
- Line 8, as John would be a knave but he tells the truth

The only options are Line 3 and 5; The treasure is on the island, but we only know that John and Sally are of opposite types.

3) *Let p and q be two propositions. Use a truth table or logical equivalence to indicate if the proposition $\neg(p \to \neg q) \to \neg(p \leftrightarrow \neg q)$ is a tautology, a contradiction, or neither*

Let us build the truth table for $A = \neg(p \to \neg q) \to \neg(p \leftrightarrow \neg q)$:

p	q	$\neg q$	$(p \to \neg q)$	$\neg(p \to \neg q)$	$p \leftrightarrow \neg q$	$\neg(p \leftrightarrow \neg q)$	A
T	T	F	F	T	F	T	T
T	F	T	T	F	T	F	T
F	T	F	T	F	T	F	T
F	F	T	T	F	F	T	T

From Column 8, we can conclude that A is a tautology.

Part III *4 problems, each 10 points; total 40 points)*

1) *Give a direct proof and a proof by contradiction of the proposition: if $2n^3 + 3n^2 + 4n + 3$ is odd, then n is even, where n is a natural number.*

Let p be the proposition "$2n^3 + 3n^2 + 4n + 3$ is odd" and q be the proposition "n is even". We want to show that $p \to q$ is true.

i) Let us show $p \to q$ using a direct proof:
Hypothesis: p is true, i.e. $2n^3 + 3n^2 + 4n + 3$ is odd. There exists an integer k such that let $2n^3 + 3n^2 + 4n + 3 = 2k + 1$,We get:

$$\begin{aligned} n^2 &= -2n^3 - 2n^2 - 4n + 2k - 2 \\ &= 2(-n^3 - n^2 - 2n + k - 1) \end{aligned}$$

As $-n^3 - n^2 - 2n + k - 1$ is an integer, n^2 is even. We know that if n^2 is even, then n is even. Therefore n is even. We have shown that q is true when p is true: $p \to q$ is true.

ii) Let us show $p \to q$ using a proof by contradiction:
We use a proof by contradiction, i.e. we assume that p is true AND $\neg q$ is true. Since $\neg q$ is true, n is odd. If n is odd, then there exists an integer k such $n = 2k + 1$. We get:

$$
\begin{aligned}
2n^3+3n^2+4n+3 &= 2(2k+1)^3+3(2k+1)^2+4(2k+1)+3 \\
&= 2(8k^3+12k^2+6k+1)+3(4k^2+4k+1)+8k+7 \\
&= 16k^3+36k^2+32k+12 \\
&= 2(8k^3+18k^2+16k+6)
\end{aligned}
$$

As $8k^3+18k^2+16k+6$ is an integer, $2n^3+3n^2+4n+3$ is even. Remember however that we had assumed that p is true, i.e. that $2n^3+3n^2+4n+3$ is odd. We have reached a contradiction, and therefore $p \to q$ is true.

2) *Use induction to prove that any postage value of n cents can be made with only 5-cent stamps and 6-cent stamps, whenever $n \geq 20$, n natural number.*

Let $p(n)$ be the proposition that n cents can be made with only 5–cent and 6–cent stamps, when n is greater than or equal to 20.
Therefore there exists two positive integers a_n and b_n such that $n = 5a_n + 6b_n$

a) Basis step: I want to prove that $p(20)$ is true
20 can be composed of 5 times 4 plus 0 times 6: $20 = 5 \times 4 + 6 \times 0$
We can set $a_{20} = 4$ and $b_{20} = 0$. Both are positive integers. Therefore $p(20)$ is true

b) Inductive Step
I want to show $p(n) \to p(n+1)$ whenever $n \geq 20$
Hypothesis: $p(n)$ is true and there exists two positive integers a_n and b_n such that $n = 5a_n + 6b_n$
Then:
$n+1 = 5a_n + 6b_n + 1$
Since 1 can be written as $6-5$ we can write
$n+1 = 5a_n + 6b_n + 6 - 5 = 5(a_n-1) + 6(b_n+1)$
Since b_n is greater than or equal to 0, then (b_n+1) is also greater than 0
(a_n-1) is only positive if a_n is greater or equal to 1.
There are therefore two situations that we need to consider: $a_n \geq 1$ and $a_n = 0$.

i) $a_n \geq 1$
Then $n+1$ can be written as:
$n+1 = 5(a_n-1) + 6(b_n+1)$ where both (a_n-1) and (b_n+1) are positive.
We can set $a_{n+1} = a_n - 1$ and $b_{n+1} = b_n + 1$. In this case, $p(n+1)$ is true.

ii) $a_n = 0$
$n+1 = 6b_n + 1$
$n+1 = 6b_n + 25 - 24$
$n+1 = 5 \times 5 + 6(b_n - 4)$
$n+1$ can be written as 5 times a positive integer 5 and 6 times (b_n-4).
Notice that $n = 6b_n$. Since $n > 20$, $6b_n > 20$. Since b_n is an integer, we conclude that $b_n \geq 4$. Therefore $(b_n-4) \geq 0$.
We can set $a_{n+1} = 5$ and $b_{n+1} = b_n - 4$. In this case, $p(n+1)$ is true

In all cases, we have proven that $p(n+1)$ is true: the inductive step is true.

The principle of proof by mathematical induction allows us to conclude that $p(n)$ is true for all
$n > 20$.

3) *Show that:*

$$\forall n \in \mathbb{N}, \quad \sum_{i=1}^{n}(-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$$

We want to Prove : $P(n)$ is true, for all $n \geq 1$, where
$P(n)$: $\displaystyle\sum_{i=1}^{n}(-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$
Let us define $LHS(n) = \displaystyle\sum_{i=1}^{n}(-1)^i i^2$, and $RHS(n) = \frac{(-1)^n n(n+1)}{2}$.

- *Basis step:* We want to prove $P(1)$ is true.
  $LHS(1) = (-1)^1 1^2 = -1$,
  and
  $RHS(1) = \frac{(-1)^1 \times 1 \times 2}{2} = -1$.
  Therefore, $LHS(1) = RHS(1)$: $P(1)$ is true.
- *Inductive step:* Let $P(n)$ be true for an integer $n \geq 1$, which means $LHS(n) = RHS(n)$. To prove that $P(n+1)$ is true, we prove that $LHS(n+1) = RHS(n+1)$. Let us compute $LHS(n+1)$:

$$\begin{aligned}
LHS(n+1) &= \sum_{i=1}^{n+1}(-1)^i i^2 \\
&= \sum_{i=1}^{n}(-1)^i i^2 + (-1)^{n+1}(n+1)^2 \\
&= LHS(n) + (-1)^{n+1}(n+1)^2 \\
&= RHS(n) + (-1)^{n+1}(n+1)^2 \\
&= \frac{(-1)^n n(n+1)}{2} + (-1)^{n+1}(n+1)^2 \\
&= \frac{(-1)^n n(n+1) + 2(-1)^{n+1}(n+1)^2}{2} \\
&= \frac{(-1)^{n+1}(n+1)(-n+2(n+1))}{2} \\
&= \frac{(-1)^{n+1}(n+1)(n+2)}{2}
\end{aligned}$$

  and

$$RHS(n+1) = \frac{(-1)^{n+1}(n+1)(n+2)}{2}$$

  Therefore $LHS(n+1) = RHS(n+1)$, i.e. $P(n+1)$ is true.

According to the principle of mathematical induction, we can conclude that $\sum_{i=1}^{n}(-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}$ for all $n \geq 1$.

4) *Let f_n be the n-th Fibonacci number (note: Fibonacci numbers satisfy $f_0 = 0$, $f_1 = 1$ and $f_n + f_{n+1} = f_{n+2}$). Prove by induction that for all natural numbers $n \geq 3$,*

$$\frac{f_1}{f_2 f_3} + \frac{f_2}{f_3 f_4} + \ldots + \frac{f_{n-2}}{f_{n-1} f_n} = 1 - \frac{1}{f_n}$$

We want to Prove : $P(n)$ is true, for all $n \geq 3$, where
$P(n)$: $\frac{f_1}{f_2 f_3} + \frac{f_2}{f_3 f_4} + \ldots + \frac{f_{n-2}}{f_{n-1} f_n} = 1 - \frac{1}{f_n}$
Let us define $LHS(n) = \frac{f_1}{f_2 f_3} + \frac{f_2}{f_3 f_4} + \ldots + \frac{f_{n-2}}{f_{n-1} f_n}$, and $RHS(n) = 1 - \frac{1}{f_n}$.

- *Basis step:* We want to prove $P(3)$ is true.
  $LHS(3) = \frac{f_1}{f_2 f_3} = \frac{1}{1 \times 2} = \frac{1}{2}$,
  and
  $RHS(3) = 1 - \frac{1}{f_3} = 1 - \frac{1}{2} = \frac{1}{2}$.

  Therefore, $LHS(3) = RHS(3)$: $P(3)$ is true.
- *Inductive step:* Let $P(n)$ be true for an integer $n \geq 3$, which means $LHS(n) = RHS(n)$. To prove that $P(n+1)$ is true, we prove that $LHS(n+1) = RHS(n+1)$. Let us compute $LHS(n+1)$:

$$\begin{aligned} LHS(n+1) &= \frac{f_1}{f_2 f_3} + \frac{f_2}{f_3 f_4} + \ldots + \frac{f_{n-2}}{f_{n-1} f_n} + \frac{f_{n-1}}{f_n f_{n+1}} \\ &= LHS(n) + \frac{f_{n-1}}{f_n f_{n+1}} \\ &= RHS(n) + \frac{f_{n-1}}{f_n f_{n+1}} \\ &= 1 - \frac{1}{f_n} + \frac{f_{n-1}}{f_n f_{n+1}} \\ &= 1 - \frac{f_{n+1} - f_{n-1}}{f_n f_{n+1}} \\ &= 1 - \frac{f_n + f_{n-1} - f_{n-1}}{f_n f_{n+1}} \\ &= 1 - \frac{f_n}{f_n f_{n+1}} \\ &= 1 - \frac{1}{f_{n+1}} \end{aligned}$$

  and

$$RHS(n+1) = 1 - \frac{1}{f_{n+1}}$$

Therefore $LHS(n+1) = RHS(n+1)$, i.e. $P(n+1)$ is true.

According to the principle of mathematical induction, we can conclude that $P(n)$ is true for all $n \geq 3$.