Data, Logic, and Computing

ECS 17 (Winter 2022)

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Midterm 2: solutions

Exercise 1 *(2 questions, 20 points total)*

Let n be an integer. Give a direct proof and an indirect proof of the proposition, if n is odd then $2n^2+5n+2$ is odd

We want to prove an implication of the form $p \rightarrow q$ is true, with:

p: n is odd

$\neg p$: n is even

q: $2n^2+5n+2$ is odd

$\neg q$: $2n^2+5n+2$ is even

We use two methods of proof:

a) Direct proof: we show $p \rightarrow q$ is true.

Let us assume that p is true, i.e. that n is odd. There exists and integer k such that $n = 2k+1$. Therefore,

$$\begin{aligned} 2n^2+5n+2 &= 2(2k+1)^2+5(2k+1)+2 \\ &= 8k^2+18k+9 \\ &= 2(4k^2+9k+4)+1 \end{aligned}$$

As k is an integer, $4k^2+9k+4$ is an integer which we call l. Therefore $2n^2+5n+2 = 2l+1$, i.e. it is odd.

We have shown that q is true when p is true: the proposition $p \rightarrow q$ is true.

b) Indirect proof: we show $\neg q \rightarrow \neg p$ is true.

Let us assume that $\neg q$ is true, i.e. that $2n^2+5n+2$ is even. There exists and integer k such that $2n^2+5n+2 = 2k$. Therefore,

$$\begin{aligned} 2n^2+4n+n+2 &= 2k \\ n &= 2k-2n^2-4n-2 \\ &= 2(k-n^2-2n-1) \end{aligned}$$

As k and n are integers, $k - n^2 - 2n - 1$ is an integer which we call l. Therefore $n = 2l$, i.e. it is even.

We have shown that $\neg p$ is true when $\neg q$ is true: the proposition $\neg q \rightarrow \neg p$ is true and, by equivalence, $p \rightarrow q$ is true.

Exercise 2 *(1 question, 10 points)*

Let m and n be 2 integers. Using the method of proof of your choice, show that if mn is odd, then m is odd and n is odd.

We want to prove an implication of the form $p \rightarrow q$ is true, with:

p: mn is odd

$\neg p$: mn is even

q: m is odd and n is odd

$\neg q$: m is even or n is even

We use an indirect proof: we show that $\neg q \rightarrow \neg p$ is true.
Let us assume that $\neg q$ is true, namely that m is even or n is even. We consider two cases:

a) m is even. There exists an integer k such that $m = 2k$. Then,

$$\begin{aligned} mn &= 2kn \\ &= 2(kn) \end{aligned}$$

As k and n are integers, kn is an integer which we call l. Therefore $mn = 2l$, i.e. it is even.

b) n is even. There exists an integer k such that $n = 2k$. Then,

$$\begin{aligned} mn &= 2km \\ &= 2(km) \end{aligned}$$

As k and m are integers, km is an integer which we call l. Therefore $mn = 2l$, i.e. it is even.

In both cases, we have shown that mn is even. Therefore $\neg p$ is true when $\neg q$ is true. the proposition $\neg q \rightarrow \neg p$ is true and, by equivalence, $p \rightarrow q$ is true.

Exercise 3 *(1 question, 10 points)*

Let n be an integer. Use a proof by contradiction to show that $\frac{6n+1}{2n+4}$ is not an integer.

Let:
P: $\frac{6n+1}{2n+4}$ is not an integer
We use a proof by contradiction. We **assume** that P is false, i.e. we assume that $\frac{6n+1}{2n+4}$ is an integer. Let us name this integer as k. We have:

$$\frac{6n+1}{2n+4} = k$$

which we rewrite as:

$$6n + 1 = k(2n + 4)$$

Let $LHS = 6n + 1$ and $RHS = k(2n + 4)$. Notice that:

$$LHS = 2(3n) + 1$$

Since n is an integer, $3n$ is an integer and therefore LHS is odd. Conversely,

$$RHS = 2(k(n + 2))$$

As k and n are integers, $k(n+2)$ is an integer which we call l. Therefore $RHS = 2l$, i.e. it is even.

Under the assumption that P is false, we find that $LHS = RHS$ with LHS odd and RHS even. Since an even number cannot be equal to an odd number, we have reached a contradiction. Therefore the assumption that P is false, is false, i.e. P is true.

Exercise 4 *(1 question, 10 points)*

Let n be a natural number (i.e., n is a positive integer different from 0). Use a proof by contradiction to show that if n is a perfect square, then 2n is not a perfect square. (A natural number n is a perfect square if and only if there exists an integer k such that $n = k^2$).

We want to prove an implication of the form $p \rightarrow q$ is true, with:

p: n is a perfect square

$\neg p$: n is not a perfect square

q: $2n$ is not a perfect square

$\neg q$: $2n$ is a perfect square

We use a proof by contradiction. We assume that $p \rightarrow q$ is false, i.e. that p is true AND q is false.

Since p is true, n is a perfect square: there exists an integer k such that $n = k^2$.

Since q is false, $2n$ is a perfect square: there exists an integer l such that $2n = l^2$.

Replacing n by k^2, we get:

$$2k^2 = l^2$$

As n is non zero, l is not zero. Therefore:

$$2 = \frac{k^2}{l^2}$$

Taking the square root (the numbers are now real),

$$\sqrt{2} = \frac{|k|}{|l|}$$

As k is an integer, $|k|$ is an integer. Similarly, as l is an integer, $|l|$ is an integer. This would lead to $\sqrt{2}$ is rational: this is a contradiction, as we know that $\sqrt{2}$ is irrational.

Therefore the assumption that $p \rightarrow q$ is false, is false, i.e. $p \rightarrow q$ is true.

Exercise 5 *(1 question, 10 points)*

Let x be a real number. Show that if $x^3 + x^2 - 2x < 0$, then $x < 1$.

We want to prove an implication of the form $p \to q$ is true, with:

p: $x^3 + x^2 - 2x < 0$

$\neg p$: $x^3 + x^2 - 2x \geq 0$

q: $x < 1$

$\neg q$: $x \geq 1$

We use an indirect proof, i.e. we prove that $\neg q \to \neg p$ is true. We assume that $\neg q$ is true, i.e. that $x \geq 1$.

Let $A = x^3 + x^2 - 2x$. Notice that,

$$\begin{aligned} A &= x^3 + x^2 - 2x \\ &= x(x-1)(x+2) \end{aligned}$$

We know that:

i) $x > 0$ since $x \geq 1$

ii) $x - 1 \geq 0$ since $x \geq 1$

iii) $x + 2 > 0$ since $x \geq 1$

The three terms in A are positive: A is positive. Therefore $\neg p$ is true.

We have shown that $\neg p$ is true when $\neg q$ is true. the proposition $\neg q \to \neg p$ is true and, by equivalence, $p \to q$ is true.

Exercise 6 *(1 question, 10 points)*

Prove or disprove that there exits an integer n such that $n^2 + 3n + 2$ is odd.

Let:
P : There exists an integer n such that $n^2 + 3n + 2$ is odd

P is likely to be false. To prove that it is false, we need to show that $\neg P$ is true, namely that

$\neg P$: For all integers n, $n^2 + 3n + 2$ is even.

We use a proof by case:

case a) n is even.

There exists an integer k such that $n = 2k$. Then,

$$\begin{aligned} n^2 + 3n + 2 &= (2k)^2 + 3(2k) + 2 \\ &= 4k^2 + 6k + 2 \\ &= 2(2k^2 + 3k + 1) \end{aligned}$$

As k is an integer, $2k^2 + 3k + 1$ is an integer which we call l. Therefore $n^2 + 3n + 2 = 2l$, i.e. it is even.

case b) n is odd.

There exists an integer k such that $n = 2k + 1$. Then,

$$\begin{aligned} n^2 + 3n + 2 &= (2k+1)^2 + 3(2k+1) + 2 \\ &= 4k^2 + 4k + 1 + 6k + 3 + 2 \\ &= 2(2k^2 + 5k + 3) \end{aligned}$$

As k is an integer, $2k^2 + 5k + 3$ is an integer which we call l. Therefore $n^2 + 3n + 2 = 2l$, i.e. it is even.

In all cases, $n^2 + 3n + 2$ is even.

We have shown that $\neg P$ is true, therefore the original proposition P is false.