Discussion 3/6

Exercise 1

Fibonacci sequence:

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2}$$

$F_0 = 0$

$F_1 = 1$

$F_2 \quad 1$

$F_3 \quad 2$

$F_4 \quad 3$

$F_5 \quad 5$

$F_6 \quad 8$

$F_7 \quad 13$

Exercise 1

Show that

$$1 + \sum_{i=0}^{n} F_{2i} = F_{2n+1}$$

Definitions:

$$LHS(n) = 1 + \sum_{i=0}^{n} F_{2i}$$

$$RHS(n) = F_{2n+1}$$

$$P(n): \quad LHS(n) = RHS(n)$$

I want to show $P(n)$ is true for $n \in \mathbb{N}$

Proof by induction

Basis step: is $P(1)$ true?

$$LHS(1) = 1 + \sum_{i=0}^{1} F_{2i} = 1 + F_0 + F_2 = 2$$

$$RHS(1) = F_{2+1} = F_3 = 2$$

Therefore $LHS(1) = RHS(1)$: $P(1)$ is true.

Exercise 3

Show that F_{4n} is a multiple of 3 for all $n \in \mathbb{N}$.

Definition:

$P(n)$: there exists an integer k such that $F_{4n} = 3k$

I want to prove $P(n)$ is true for all $n \in \mathbb{N}$

I use a proof by induction.

Basis step: is $P(1)$ true?

$$F_4 = F_3 + F_2 = F_2 + F_1 + F_1 + F_0$$
$$= F_1 + F_0 + F_1 + F_1 + F_0$$
$$= 3$$

F_4 is a multiple of 3 : $P(1)$ is true.

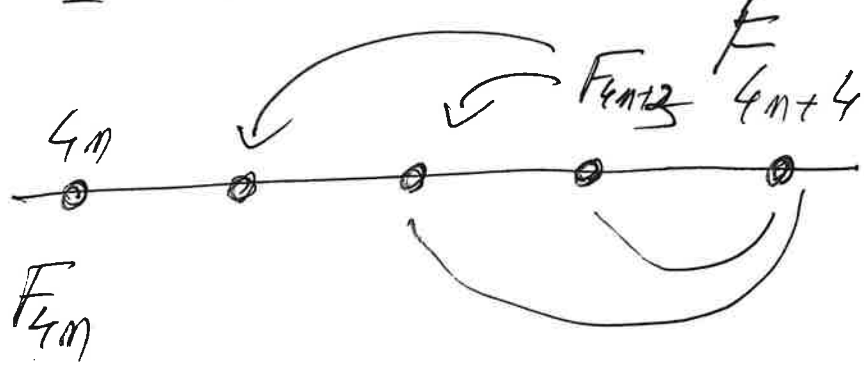
Inductive step: $P(n) \to P(n+1)$, $n \in \mathbb{N}$. 5

I assume $P(n)$ is true:

there exists an integer k such that

$$F_{4n} = 3k$$

F_{4n+4} ?



$$
\begin{aligned}
F_{4n+4} &= F_{4n+3} + F_{4n+2} \\
&= F_{4n+2} + F_{4n+1} + F_{4n+1} + F_{4n} \\
&= F_{4n+1} + F_{4n} + 2F_{4n+1} + F_{4n} \\
&= 3F_{4n+1} + 2F_{4n} \\
&= 3F_{4n+1} + 2(3k) \\
&= 3\big[\underbrace{F_{4n+1} + 2k}_{\text{integer}}\big]
\end{aligned}
$$

F_{4n+4} is a multiple of 3: $P(n+1)$ is true.

The method of proof by induction allows me to conclude that $P(n)$ is true for all $n \in \mathbb{N}$.

Exercise 3 : Let a_n be the sequence

$$a_1 = 1$$
$$a_n = a_{n-1} + 2n - 1 \quad n \geq 2.$$

Can you find a closed form?

$$a_1 = 1 = 1^2$$
$$a_2 = 1 + 2\times2 - 1 = 4 = 2^2$$
$$a_3 = 4 + 6 - 1 = 9 = 3^2$$
$$a_4 = 9 + 8 - 1 = 16 = 4^2$$

$P(n)$: $\boxed{a_n = n^2}$

Definitions:

$$LHS(n) = a_n$$
$$RHS(n) = n^2$$
$$P(n): \; LHS(n) = RHS(n)$$

Proof by induction:

Basis step: is $P(1)$ true?

$LHS(1) = a_1 = 1$

$RHS(1) = 1^2 = 1$

Therefore $LHS(1) = RHS(1)$; $P(1)$ is true.

Inductive step: $P(n) \rightarrow P(n+1) \quad n \in \mathbb{N}$.

I assume $P(n)$ is true: $a_n = n^2$

$$\begin{aligned} LHS(n+1) &= a_{n+1} \\ &= a_n + 2(n+1) - 1 \\ &= n^2 + 2n + 2 - 1 \\ &= n^2 + 2n + 1 = (n+1)^2 \end{aligned}$$

$RHS(n+1) = (n+1)^2$

Therefore $LHS(n+1) = RHS(n+1)$: $P(n+1)$ is true

The method of proof by induction allows me to conclude that $a_n = n^2$ for all $n \in \mathbb{N}$.