Discrete Mathematics

ECS 20 (Winter 2019)

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Discussion 1 - 1/09-1/15

Exercise 1

Let a and b be two real numbers.

a) Show that $(a^2 + b^2)^2 = (a^2 - b^2)^2 + (2ab)^2$
Let $LHS = (a^2 + b^2)^2$ and $RHS = (a^2 - b^2)^2 + (2ab)^2$. Then:

$$LHS = a^4 + b^4 + 2a^2b^2$$

and

$$\begin{aligned} RHS &= a^4 + b^4 - 2a^2b^2 + 4a^2b^2 \\ &= a^4 + b^4 + 2a^2b^2 \end{aligned}$$

Therefore $LHS = RHS$ for all a and b, and the identity is true.

b) $a^4 - b^4 = (a - b)(a + b)(a^2 + b^2)$
Let $LHS = a^4 - b^4$ and $RHS = (a - b)(a + b)(a^2 + b^2)$. Then:

$$\begin{aligned} RHS &= (a^2 - b^2)(a^2 + b^2) \\ &= a^4 - b^4 \end{aligned}$$

Therefore $LHS = RHS$ for all a, b, and the identity is true.

Exercise 2

a) Show that there are no positive integer number n such that $n^2 + n^3 = 100$

Let n be a positive integer. Since $n >= 0$, $n^2 >= 0$ and $n^3 >= 0$. We note first that if $n \geq 5$, then $n^3 \geq 125$, i.e. $n^3 > 100$, and therefore $n^2 + n^3 > 100$. The only possible solutions are therefore $n = 0$, $n = 1$, $n = 2$, $n = 3$, and $n = 4$. We test each of those values separately:

i) $n = 0$ then $n^2 + n^3 = 0 \neq 100$. $n = 0$ is not a solution.

ii) $n = 1$ then $n^2 + n^3 = 2 \neq 100$. $n = 1$ is not a solution.

iii) $n = 2$ then $n^2 + n^3 = 12 \neq 100$. $n = 2$ is not a solution.

iv) $n = 3$ then $n^2 + n^3 = 36 \neq 100$. $n = 3$ is not a solution.

v) $n = 4$ then $n^2 + n^3 = 80 \neq 100$. $n = 4$ is not a solution.

Therefore there are no positive integer number n such that $n^2 + n^3 = 100$.

b) Prove that there are no solutions in integers x and y to the equation $2x^2 + 5y^2 = 14$.

Let x and y be two integers. We note first that $x^2 \geq 0$ and $y^2 \geq 0$. Then, if $y \leq -2$ or $y \geq 2$, $y^2 \geq 4$ and $5y^2 \geq 20$. Therefore we can conclude that $y = -1$, $y = 0$, or $y = 1$. We look at all three cases separately:

i) $y = -1$ or $y = 1$; then $2x^2 = 9$; the left hand side is even, while the right hand side is odd: this equation has no solution.

ii) $y = 0$ then $2x^2 = 14$ or $x^2 = 7$. We check all possible values of x:

- $x = 0$; then $x^2 = 0 \rightarrow$ No.
- $x = 1$ or $x = -1$; then $x^2 = 1 \rightarrow$ No.
- $x = 2$ or $x = -2$; then $x^2 = 4 \rightarrow$ No.
- $x \geq 3$ or $x < leq - 3$ then $x^2 \geq 9 \rightarrow$ No.

Therefore there are no integers x and y that satisfy the equation $2x^2 + 5y^2 = 14$.

Exercise 3

Let x be a real number. Solve $\sqrt{x^2 - 7} = \sqrt{1 - x^2}$

We need to define the domain of the equation first. This equation involves two square root functions that are defined if and only if their arguments are positive. Therefore: $D = \{x \in \mathbb{R} \quad | \quad x^2 - 7 \geq 0 \quad and \quad 1 - x^2 \geq 0\}$.

Let us look at both conditions:
i) $x^2 - 7 \geq 0$ implies that $x \leq \sqrt{7}$ or $x \geq \sqrt{7}$
ii) $1 - x^2 \geq 0$ implies that $-1 \leq x \leq 1$
These two conditions are incompatible. Therefore $D = \emptyset$, and the equation does not have any solutions.

Exercise 4

Three consecutive integers add up to 51. What are those three integers?

Let a be an integer. The two integers that follow a are $a+1$ and $a+2$. Therefore:

$$\begin{aligned} a + a + 1 + a + 2 &= 51 \\ 3a + 3 &= 51 \\ 3a &= 48 \\ a &= 16 \end{aligned}$$

Therefore the three consecutive integers that add up to 51 are 16, 17, and 18.