Review session: Solutions

ECS 20 (Fall 2016)

Patrice Koehl
koehl@cs.ucdavis.edu

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1 Simple propositions

For each proposition on the left, indicate if it is a tautology or not:

Table 1: Propositional logic

Proposition	Tautology (Yes/ No)
$(\neg(p \wedge q)) \leftrightarrow (\neg p \vee \neg q)$	Yes: this is one of DeMorgan's laws
$(\neg(p \wedge q)) \leftrightarrow (\neg p \wedge \neg q)$	No! contradicts DeMorgan's law
$(\neg(p \vee q)) \leftrightarrow (\neg p \wedge \neg q)$	Yes: this is the second DeMorgan's law
if $6^2 = 36$ then $2 = 3$	No: p is true and q is false: therefore $p \rightarrow q$ is false
if $6^2 = 36$ then $gcd(10, 5) = 5$	Yes: p is true and q is true: therefore $p \rightarrow q$ is true
if $6^2 = -1$ then $36 = -1$	Yes: p is false and therefore $p \rightarrow q$ is always true.

2 Knights and Knaves

A very special island is inhabited only by Knights and Knaves. Knights always tell the truth, while Knaves always lie. You meet three inhabitants: Alex, John and Sally. Alex says, "If John is a Knight then Sally is a Knight". John says, "Alex is a Knight and Sally is a Knave". Can you find what Alex, John, and Sally are? Explain your answer.

Let us build the table for the possible options for Alex, John, and Sally. We then check the validity of the two statements, and finally check the consistency of the truth values for those statements with the nature of Alex and John.

Line	Alex	John	Sally	Alex says	John says	Compatibility
1	Knight	Knight	Knight	T	F	No: John would be a Knight who lies
2	Knight	Knight	Knave	F	T	No: Alex would be a Knight who lies
3	Knight	Knave	Knight	T	F	Yes
4	Knight	Knave	Knave	T	T	No, John would be a Knave who tells the truth
5	Knave	Knight	Knight	T	F	No, Alex would be a Knave who tells the truth
6	Knave	Knight	Knave	F	F	No, John would be a Knight who lies
7	Knave	Knave	Knight	T	F	No, Alex would be a Knave who tells the truth
8	Knave	Knave	Knave	T	F	No, Alex would be a Knave who tells the truth

Therefore Alex and Sally are Knights and John is a Knave.

3 Proofs: direct, indirect, and contradictions

3.1 Different methods of proofs

Let n be an integer. Show that if $3n^2 + 2n + 9$ is odd, then n is even using a direct, indirect, and proof by contradiction.

This is a problem of showing a conditional $p \to q$ is true, where
$p : 3n^2 + 2n + 9$ is odd
$q : n$ is even

We will use two different types of proof: direct, and proof by contradiction

a) Direct proof: we show directly that $p \to q$ is true.

   Hypothesis: p is true, $3n^2 + 2n + 9$ is odd. Therefore there exists an integer k such that $3n^2 + 2n + 9 = 2k + 1$, i.e. $3n^2 = 2k - 2n - 8 = 2(k - n - 4)$. Therefore 2 divides $3n^2$. Since 2 is a prime number, according to Euclid's theorem, we conclude that 2 divides 3 or 2 divides n; since 2 does not divide 3, we conclude that 2 divides n, therefore n is even. We have showed that q is true, therefore $p \to q$ is true

b) Proof by contradiction: we suppose $p \to q$ is false

   Hypothesis: $p \to q$ is false, i.e. p is true and $\neg q$ is true, namely $3n^2 + 2n + 9$ is odd and n is odd.

   Since n is odd, there exists an integer k such that $n = 2k + 1$. Therefore, $3n^2 + 2n + 9 = 3(2k + 1)^2 + 2(2k + 1) + 9 = 12k^2 + 16k + 14 = 2(6k^2 + 8k + 7)$

   Since $6k^2 + 8k + 7$ is integer, $3n^2 + 2n + 9$ is even. But we have supposed that $3n^2 + 2n + 9$ is odd. We have reached a contradiction. Therefore the hypothesis we made is false, therefore $p \to q$ is true.

3.2 Proof by contradiction

Let n be a strictly positive integer. Show that $\frac{6n+1}{2n+4}$ is not an integer

We use a proof by contradiction: We make the hypothesis that $\frac{6n+1}{2n+4}$ is an integer. Let us write this integer as k. Then we have:
$6n + 1 = k(2n + 4) = 2k(n + 2)$
This would mean however that an odd number, $6n + 1$, is equal to an even number, $2k(n + 2)$. This is a contradiction. Therefore the hypothesis is wrong and the property, namely $\frac{6n+1}{2n+4}$ is not an integer, is true.

3.3 Proof by contradiction

Let n be a strictly positive integer. Show that if $\sqrt{n^2+1}$ is not an integer.

We use a proof by contradiction: We make the hypothesis that $\sqrt{n^2+1}$ is an integer. Let us write this integer as k. Then we have:

$$\begin{aligned}
\sqrt{n^2+1} &= k \\
n^2+1 &= k^2 \\
k^2-n^2 &= 1 \\
(k-n)(k+n) &= 1
\end{aligned}$$

Since k and n are supposed to be integers, there are only two possibilities:

a) $k - n = 1$ and $k + n = 1$, in which case $k = 1$ and $n = 0$.

b) $k - n = -1$ and $k + n = -1$, in which case $k = -1$ and $n = 0$.

In both cases, we have $n = 0$. However, n is set to be strictly positive. We have reached a contradiction, and therefore $\sqrt{n^2+1}$ is not an integer.

3.4 Number theory: gcd

Let a and b be two strictly positive integers. Show that if $\gcd(a, b) = 1$ then $\gcd(a, b^2) = 1$.

This is a problem of showing a conditional $p \to q$ is true, where
$p : \gcd(a, b) = 1$
$q : \gcd(a, b^2) = 1$

We will use a direct proof.

Hypothesis: p is true, $\gcd(a, b) = 1$. Let us define $g = \gcd(a, b^2)$. We want to show that $g = 1$.
According to Bezout's identity, there exist two integers m and n such that:
$am + bn = 1$
After multiplication by b, we get:
$abm + b^2n = b$
Since $g = \gcd(a, b^2)$, there exits two integers u and v such that $a = gu$ and $b^2 = gv$. Replacing in

the equation above, we get:
$gubm + gvn = b$, i.e.
$g(ubm + vn) = b$.
Therefore g/b. By definition of g, we also have g/a. Therefore g is a common divisor of a and b, and since the only common divisor of a and b is 1, we conclude that $g = 1$, which concludes the proof.

3.5 Number theory: gcd

Let a and b be two strictly positive integers. Show that if $\gcd(a, b) = 1$ then $\gcd(a^2, b^2) = 1$ using a proof by contradiction.

This is a problem of showing a conditional $p \to q$ is true, where
$p : \gcd(a, b) = 1$
$q : \gcd(a^2, b^2) = 1$

We will use a proof by contradiction.

Hypothesis: $p \to q$ is false. i.e. p is true and q is false, i.e. $\gcd(a, b) = 1$ and $\gcd(a^2, b^2) > 1$. Let us define $g = \gcd(a^2, b^2)$. Since $g > 1$, g can be decomposed as a product of prime numbers. Let d be one of those prime numbers. Since d/g, d/a^2 and d/b^2. Since d is prime, based on Euclid's theorem, we have d/a and d/b; this contradict however that $\gcd(a, b) = 1$. Therefore the hypothesis that $p \to q$ is false is false, i.e. $p \to q$ is true.

3.6 Number theory: divisibility

Let a and b be two strictly positive integers with $\gcd(a, b) = 1$. Let c be another strictly positive integers. Show that if a/c and b/c, then ab/c.

Let a and b be two strictly positive integers with $\gcd(a, b) = 1$. We need to show the conditional $p \to q$ is true, where
$p : a/c$ and b/c
$q : ab/c$

We will use a direct proof.

Hypothesis: p is true, namely a/c and b/c. Therefore there exist two integers m and n such that $c = am$ and $c = bn$. We also know that $\gcd(a, b) = 1$. According to Bezout's identity, there exist two integers u and v such that:
$au + bv = 1$
After multiplication by c, we get:
$acu + bcv = c$
Replacing c in the first term and second term on the left by bn and am, respectively, we find:
$abnu + abmv = c$
$ab(nu + mv) = c$
Therefore ab/c: the implication $p \to q$ is true.

4 Functions

4.1 Function floor

Let n be an integer and x a real number. Show that $n \leq x$ if and only if $n \leq \lfloor x \rfloor$.

Let $p : n \leq x$
and let $q : n \leq \lfloor x \rfloor$
Showing that the biconditional $p \leftrightarrow q$ is true means that we must prove p implies q and q implies p.

Case 1: $p \rightarrow q$
Hypothesis: p is true, therefore $n \leq x$
From the definition of floor, $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$
Since n is smaller than x, $\lfloor x \rfloor$ is smaller than x, and by definition $\lfloor x \rfloor$ is the largest integer smaller than x, we conclude that $n \leq \lfloor x \rfloor$, and therefore that q is true.

Case 2: $q \rightarrow p$
Hypothesis: q is true, therefore $n \leq \lfloor x \rfloor$
From the definition of floor, $\lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$
We have $n \leq \lfloor x \rfloor$ and $\lfloor x \rfloor \leq x$, therefore $n \leq x$ and therefore $q \rightarrow p$ is also true.
Since $p \rightarrow q$ is true and $q \rightarrow p$ is true, the biconditional is true.

4.2 Function floor

Let x be a real number. Show that the integer $\lfloor x + \frac{1}{2} \rfloor + \lfloor x - \frac{1}{2} \rfloor$ is odd.

I will solve this problem using two different methods: there might be one or the other that you feel more comfortable with!

a) Method 1:
Let us define $n = \lfloor x \rfloor$ and let us write $x = n + \epsilon$, where $0 \leq \epsilon < 0$. Then $x + \frac{1}{2} = n + \epsilon + \frac{1}{2}$ and $x - \frac{1}{2} = n + \epsilon - \frac{1}{2}$. Then,

$$\begin{aligned} \lfloor x + \frac{1}{2} \rfloor &= n + \lfloor \epsilon + \frac{1}{2} \rfloor \\ \lfloor x - \frac{1}{2} \rfloor &= n + \lfloor \epsilon - \frac{1}{2} \rfloor \end{aligned}$$

We need to distinguish two cases: $0 \leq \epsilon < \frac{1}{2}$ and $\frac{1}{2} \leq \epsilon < 1$:

i) $0 \leq \epsilon < \frac{1}{2}$
Then:

$$\begin{aligned} \frac{1}{2} \leq \epsilon + \frac{1}{2} < 1 \\ -\frac{1}{2} \leq \epsilon - \frac{1}{2} < 0 \end{aligned}$$

Then:

$$\begin{aligned}\lfloor x + \frac{1}{2} \rfloor &= n \\ \lfloor x - \frac{1}{2} \rfloor &= n - 1\end{aligned}$$

and,

$$\lfloor x + \frac{1}{2} \rfloor + \lfloor x - \frac{1}{2} \rfloor = 2n - 1$$

Therefore $\lfloor x + \frac{1}{2} \rfloor + \lfloor x - \frac{1}{2} \rfloor$ is odd.

ii) $\frac{1}{2} \leq \epsilon < 1$
Then:

$$\begin{aligned}1 \leq \epsilon + \frac{1}{2} < \frac{3}{2} \\ 0 \leq \epsilon - \frac{1}{2} < \frac{1}{2}\end{aligned}$$

Then:

$$\begin{aligned}\lfloor x + \frac{1}{2} \rfloor &= n + 1 \\ \lfloor x - \frac{1}{2} \rfloor &= n\end{aligned}$$

and,

$$\lfloor x + \frac{1}{2} \rfloor + \lfloor x - \frac{1}{2} \rfloor = 2n + 1$$

Therefore $\lfloor x + \frac{1}{2} \rfloor + \lfloor x - \frac{1}{2} \rfloor$ is odd.

In all cases, $\lfloor x + \frac{1}{2} \rfloor + \lfloor x - \frac{1}{2} \rfloor$ is odd.

b) Method 2:
Let us define: $f(x) = \lfloor x + \frac{1}{2} \rfloor + \lfloor x - \frac{1}{2} \rfloor$. Notice that:

$$\begin{aligned}f(x + \frac{1}{2}) &= \lfloor x \rfloor + 1 + \lfloor x \rfloor \\ &= 2\lfloor x \rfloor + 1\end{aligned}$$

Let $g(x) = f(x + \frac{1}{2})$. Since $f(x) = g(x - \frac{1}{2})$, we get that $f(x) = 2\lfloor x - \frac{1}{2} \rfloor + 1$, which is odd!

4.3 Function floor

Find the remainder of the division of 90^{1000} by 11.

Let $A = 90^{1000}$. Note first that $1000 = 11 \times 90 + 10$, therefore $A = (90^{90})^{11} \times 90^10$.

Since 11 is prime, we can use Fermat's little theorem, i.e. for all natural number a,

$$a^{11} \equiv a[11]$$

Therefore,

$$\begin{aligned} A &\equiv (90^{90})^{11} \times 90^{10}[11] \\ &\equiv 90^{90} \times 90^{10}[11] \\ &\equiv 90^{100}[11] \end{aligned}$$

Note now that $100 = 11 \times 9 + 1$. Then $90^{100} = (90^9)^11 \times 90$.
Therefore,

$$\begin{aligned} A &\equiv (90^9)^{11} \times 90^1[11] \\ &\equiv 90^{10}[11] \end{aligned}$$

Note now:

$$\begin{aligned} 90 &\equiv 2[11] \\ 90^2 &\equiv 4[11] \\ 90^4 &\equiv 5[11] \\ 90^8 &\equiv 3[11] \end{aligned}$$

Therefore:

$$90^{10} \equiv 1[11]$$

Therefore the remainder of the division of 90^{1000} by 11 is 1.

5 Proofs by induction

5.1 Identity

a) Show that $1 + 3 + \ldots 2n - 1 = n^2$, for all $n \geq 1$.

Let us define $LHS(n) = 1 + 3 + \ldots 2n - 1$
and $RHS(n) = n^2$
Let $p(n) : LHS(n) = RHS(n)$
We want to show $p(n)$ is true for all $n \geq 1$

a) Base Case n=1
$LHS(1) = 1$
$RHS(1) = 1^2 = 1$
Since $LHS(1) = RHS(1)$, $p(1)$ is true

b) Inductive Step
I want to show p(k)→p(k+1) whenever k≥1
Hypothesis: p(k) is true and LHS(k)=RHS(k)

$$
\begin{aligned}
LHS(k+1) &= 1 + 3 + \ldots 2n - 1 + 2n + 1 \\
&= LHS(k) + 2n + 1 \\
&= RHS(k) + 2n + 1 \\
&= n^2 + 2n + 1 \\
&= (n+1)^2 \\
&= RHS(k+1)
\end{aligned}
$$

Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 1$.

b) Show that $\displaystyle\sum_{k=1}^{n} \frac{1}{4k^2 - 1} = \frac{n}{2n+1}$ for all integer $n \geq 1$.

Let us define $LHS(n) = \sum_{k=1}^{n} \frac{1}{4k^2-1}$
and $RHS(n) = \frac{n}{2n+1}$
Let $p(n) : LHS(n) = RHS(n)$
We want to show $p(n)$ is true for all $n \geq 1$

a) Base Case n=1
$LHS(1) = \frac{1}{3}$
$RHS(1) = \frac{1}{2 \times 1 + 1} = \frac{1}{3}$
Since $LHS(1) = RHS(1)$, $p(1)$ is true

b) Inductive Step
I want to show p(k)→p(k+1) whenever k≥1
Hypothesis: p(k) is true and LHS(k)=RHS(k)

$$
\begin{aligned}
LHS(k+1) &= \sum_{i=1}^{k+1} \frac{1}{4i^2-1} \\
&= LHS(k) + \frac{1}{4(k+1)^2-1} \\
&= \frac{k}{2k+1} + \frac{1}{(2k+1)(2k+3)} \\
&= \frac{k(2k+3)+1}{(2k+1)(2k+3)} \\
&= \frac{2k^2+3k+1}{(2k+1)(2k+3)} \\
&= \frac{(2k+1)(k+1)}{(2k+1)(2k+3)} \\
&= \frac{k+1}{2k+3} \\
&= RHS(k+1)
\end{aligned}
$$

Therefore $LHS(k+1) = RHS(k+1)$, which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 1$.

5.2 Divisibility

a) Show that $5/(7^n - 2^n)$ for all integer $n \geq 1$.

Let us define $LHS(n) = 7^n - 2^n$
Let $p(n) : 5/LHS(n)$
We want to show $p(n)$ is true for all $n \geq 1$

a) Base Case n=1
$LHS(1) = 7 - 2 = 5$
Since $5/LHS(1)$, $p(1)$ is true

b) Inductive Step
I want to show p(k)→p(k+1) whenever k≥1
$p(k)$ is true means there exists an integer m such that $LHS(k) = 7^k - 2^k = 5m$.
Note that:

$$\begin{aligned}
LHS(k+1) &= 7^{k+1} - 2^{k+1} \\
&= 7 \times 7^k - 2 \times 2^k \\
&= 7 \times (5m + 2^k) - 2 \times 2^k \\
&= 5(7m) + 5 \times 2^k \\
&= 5(7m + 2^k)
\end{aligned}$$

Therefore $5/LHS(k+1)$ which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 1$.

b) Show that $6/[n(2n+1)(7n+1)]$ for all integer $n \geq 1$.

Let us define $LHS(n) = n(2n+1)(7n+1)$
Let $p(n) : 6/LHS(n)$
We want to show $p(n)$ is true for all $n \geq 1$

a) Base Case n=1
$LHS(1) = 1 \times (3) \times (8) = 24 = 6 \times 4$
Since $6/LHS(1)$, $p(1)$ is true

b) Inductive Step
I want to show p(k)→p(k+1) whenever k≥1
$p(k)$ is true means there exists an integer m such that $LHS(k) = k(2k+1)(7k+1) = 6m$.
Note that:

$$\begin{aligned}
LHS(k+1) &= (k+1)(2k+3)(7k+8) \\
&= (2k^2 + 5k + 3)(7k+8) \\
&= 14k^3 + 51k^2 + 61k + 24
\end{aligned}$$

Note also that

$$\begin{aligned}
LHS(k) &= (2k^2 + k)(7k+1) \\
&= 14k^3 + 9k^2 + k
\end{aligned}$$

Therefore:

$$\begin{aligned}
LHS(k+1) &= LHS(k) + 42k^2 + 60k + 24 \\
&= 6m + 6(7k^2 + 10k + 4) \\
&= 6(m + 7k^2 + 10k + 4)
\end{aligned}$$

Therefore $6/LHS(k+1)$ which validates that $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 1$.

5.3 Stamps

Use induction to prove that any postage of n cents (with $n \geq 18$) can be formed using only 3–cent and 10–cent stamps.

Let p(n) be the proposition that n cents can be made with only 3-cent and 10-cent stamps, when n is greater than 18.

Therefore there exists two positive integers a and b such that n=3a + 10b

a) Base Case n=18
   18 can be composed of 3 times 6 plus 0 times 10
   Therefore p(18) is true

b) Inductive Step
   I want to show p(k)→p(k+1) whenever $k \geq 18$
   Hypothesis: p(k) is true and there exists two positive integers a and b such that $k = 3a + 10b$
   $k + 1 = 3a + 10b + 1$
   Since 1 can be written as $10 - 3 \times 3$ we can write
   $k + 1 = 3a + 10b + 10 - 3 \times 3 = 3(a - 3) + 10(b + 1)$
   Since b is greater than or equal to 0, then $(b + 1)$ is also greater than 0
   $(a - 3)$ is only positive if a is greater or equal to 3.
   There are therefore four situations that we need to consider: $a \geq 3$, $a = 2$, $a = 1$, and $a = 0$.

   i) a≥3
      Then $k + 1$ can be written as:
      $k + 1 = 3(a - 3) + 10(b + 1)$ where both $(a - 3)$ and $(b + 1)$ are positive
      $p(k + 1)$ is true.

   ii) $a = 2$
      $k + 1 = 10b + 7$
      $k + 1 = 10b + 27 - 20$
      $k + 1 = 10(b - 2) + 3 \times 9$
      $k + 1$ can be written as 3 times a positive integer 9 and 10 times (b-2).
      Notice that $k = 10b + 6$. Since $k > 17$, $10b + 6 > 17$, and therefore $10b > 11$. Since b is an integer, we conclude that $b \geq 2$. Therefore $(b - 2) \geq 0$.
      Therefore p(k+1) is true.

   iii) $a = 1$
      $k + 1 = 10b + 4$
      Since $4 = 24 - 20 = 3 \times 8 - 2 \times 10$ we can write
      $k + 1 = 10(b - 2) + 3 \times 8$
      $k + 1$ can be written as 3 times a positive integer 8 and 10 times (b-2).

Notice that $k = 10b + 3$. Since $k > 17$, $10b > 14$. Since b is an integer, we conclude that $b > 2$ and therefore $b - 2 \geq 0$.
Therefore $p(k+1)$ is true.

iv) $a = 0$
$k + 1 = 10b + 1$
Since $1 = 21 - 20 = 3 \times 7 - 2 \times 10$ we can write
$k + 1 = 10(b - 2) + 3 \times 7$
$k + 1$ can be written as 3 times a positive integer 7 and 10 times (b-2).
Notice that $k = 10b$. Since $k > 17$, $10b > 17$. Since b is an integer, we conclude that $b > 2$ and therefore $b - 2 \geq 0$.
Therefore $p(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that P(n) is true for all $n > 23$.

5.4 Other

Prove by induction that for all $n \geq 1$, there exist two strictly positive integers a_n and b_n such that $(1+\sqrt{2})^n = a_n + b_n\sqrt{2}$.

Let p(n) be the proposition that there exist two strictly positive integers a_n and b_n such that $(1+\sqrt{2})^n = a_n + b_n\sqrt{2}$ for all $n \geq 1$.
We want to show $p(n)$ is true for all $n \geq 1$

a) Base Case n=1
Note that $(1+\sqrt{2}) = 1 + 1 \times \sqrt{2}$. Setting $a_1 = 1$ and $b_1 = 1$, we have $(1+\sqrt{2}) = a_1 + b_1\sqrt{2}$
Therefore p(1) is true

b) Inductive Step
I want to show p(k)→p(k+1) whenever $k \geq 1$
Hypothesis: p(k) is true and there exists two positive integers a_k and b_k such that $(1+\sqrt{2})^k = a_k + b_k\sqrt{2}$
Then,

$$\begin{aligned}(1+\sqrt{2})^{k+1} &= (1+\sqrt{2})^k(1+\sqrt{2}) \\ &= (a_k + b_k\sqrt{2})(1+\sqrt{2}) \\ &= a_k + 2b_k + (a_k + b_k)\sqrt{2}\end{aligned}$$

Let us set $a_{k+1} = a_k + 2b_k$ and $b_{k+1} = a_k + b_k$. We note first that a_{k+1} and b_{k+1} are strictly positive. Second, we have:

$$(1+\sqrt{2})^{k+1} = a_{k+1} + b_{k+1}\sqrt{2}$$

Therefore $P(k+1)$ is true.

The principle of proof by mathematical induction allows us to conclude that $P(n)$ is true for all $n \geq 1$.

5.5 Fibonacci

Let f_n be the Fibonacci numbers. show that $f_{n-1}f_{n+1} - f_n^2 = (-1)^n$, for all $n > 1$.

Let me define LHS(n)=$f_{n-1}f_{n+1} - f_n^2$
Let me define RHS(n)=$(-1)^n$
Let p(n): LHS(n)=RHS(n)
I want to show p(n) is true for all $n > 1$

a) Base Case n=2
   LHS(2)=$f_1 f_3$ - f_2^2 = (1)(2)-(1)^2=(2)-(1)=1
   RHS(2)=$(-1)^2$=1
   Since LHS(2)=RHS(2), p(2) is true

b) Inductive Step
   I want to show p(k) implies p(k+1) whenever $k > 1$
   Hypothesis: p(k) is true and LHS(k)=RHS(k)
   LHS(k+1)= $f_k f_{k+2} - f_{k+1}^2$
   LHS(k+1)= $f_k(f_k + f_{k+1}) - f_{k+1}^2$
   LHS(k+1)= $f_k^2 + f_k f_{k+1} - f_{k+1}^2$
   LHS(k+1)= $f_k^2 + f_{k+1}(f_k - f_{k+1})$
   Since $f_{k-1} + f_k = f_{k+1}$ then $f_k - f_{k+1} = -f_{k-1}$
   LHS(k+1)= $f_k^2 + f_{k+1}(-f_{k-1})$
   LHS(k+1)=$f_k^2 - f_{k+1}f_{k-1}$
   Since LHS(k)= $f_{k-1}f_{k+1} - f_k^2$
   LHS(k+1)= -LHS(k)
   Since LHS(k)=RHS(k)
   LHS(k+1)=-RHS(k) = $(-1)^{k+1}$
   RHS(k+1)=($(-1)^{k+1}$
   Therefore LHS(k + 1) = RHS(k + 1), which validates that P(k + 1) is true.

The principle of proof by mathematical induction allows us to conclude that P(n) is true for all $n > 1$.

6 Counting

6.1 Bitstrings

a) How many bit strings of length n can we form that contain at least one 0 and one 1?

There are 2^n bit strings of length n. Let S be the set of those bit strings that contain at least one 0 and one 1.
Then the complement of S, $\overline{S}$, is the set of those bit strings that do not contain one 0 or do not contain one 1. There are 1 of each, therefore $|\overline{S}| = 2$. Therefore $|S| = 2^n - 2$, and there are $2^n - 2$ bit strings of length n that contain at least one 0 and one 1.

b) How many bit strings of length 4 do not contain three consecutive 0s?

The easiest approach is to build a tree:

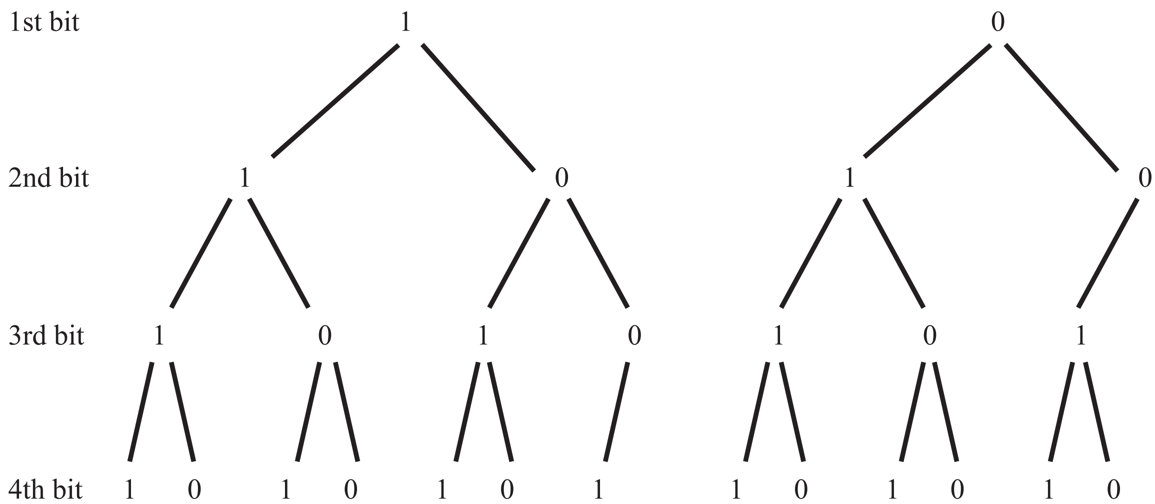

Figure 1: Bit strings of length 4 with no 3 consecutive 0s

There are therefore 13 bit strings of length 4 that do not contain three consecutive 0s.

6.2 Anagrams

a) How many words can you form with the letters of the word PATRICE?

Anagrams are simply new words formed with the same letters as the original word. Those anagrams are of length 7, with 7 options for the first letter, 6 options for the second letter, ..., 2 options for the 6-th letter, and one option for the last letter. Therefore the total number of anagrams of the word PATRICE is $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 7! = 5040$.

b) How many words can you form with the letters of the word PATRICE that start with a consonant and end with a vowel.

We proceed the same way, except that we first fill in the first and last letters. There are 4 consonants in PATRICE, therefore there are 4 possibilities for the 1st letter. There are 3 vowels in PATRICE, therefore there are 3 possibilities for the 7-th letter. Now we fill the remaining letters, from position 2 to 6: there are 5 possibilities for position 2, 4 possibilities for position 3, 3 possibilities for position 4, 2 possibilities for position 5, and 1 possibility for position 6. Therefore the total number of anagrams of the word PATRICE that start with a consonant and end with a vowel is $4 \times 5 \times 4 \times 3 \times 2 \times 1 \times 3 = 12 \times 5! = 1440$.

6.3 Subsets

Let E be a set with n elements and let A be a subset of E with p elements. How many subsets of E contain three, and only three elements of A (among possibly other elements of E)? (Reminder: there are $\dfrac{p(p-1)(p-2)}{6}$ subsets of 3 elements in a set with p elements.)

We are looking for a list of subsets of E. Those subsets have the property that they contain exactly three elements of A, as well as possibly other elements of E. Let S be one of those subsets. It can be written as:
$S = C \cup D$
where C is a subset of A that contains exactly 3 elements, and D is a subset of E that does not contain any elements of A, i.e. D is a subset of $E - A$.
To count how many ways we can build S, we have to think that we have two boxes:
- one box filled with elements of C: since we want exactly three elements of A, and A contains p elements, we have $\frac{p(p-1)(p-2)}{6}$ ways to fill this box
- one box filled with elements of D: since D is a subset of $E - A$, and there are $n - p$ elements in $E - A$, there are 2^{n-p} ways to fill that box
Using the product rule, there are $\frac{p(p-1)(p-2)}{6} \times 2^{n-p} = 2^{n-p}\frac{p(p-1)(p-2)}{6}$ subsets of E that contain exactly two elements of A.

6.4 Words

Let $A = \{\alpha, \beta, \gamma\}$ be a set with three elements. We call α, β, and γ "letters". How many words of length 4 can we form with only letters from A that contain at least one of each letter from A?

Let W be the set of words of length 4 formed with only letters from A. There are $3^4 = 81$ such words. Let B be the subset of those words that contain at least one of each letter. To find the cardinality of B, we consider instead $\overline{B}$, i.e. the subset of words that either do not contain α, or do not contain β, or do not contain γ: $\overline{B} = \overline{B}_\alpha \cup \overline{B}_\beta \cup \overline{B}_\gamma$.

- $|\overline{B}_a| = 2^4 = 16$
- $|\overline{B}_b| = 2^4 = 16$
- $|\overline{B}_c| = 2^4 = 16$
- $|\overline{B}_a \cap \overline{B}_b| = 1$
- $|\overline{B}_a \cap \overline{B}_c| = 1$
- $|\overline{B}_b \cap \overline{B}_c| = 1$
- $|\overline{B}_a \cap \overline{B}_b \cap \overline{B}_c| = 0$

Therefore $|\overline{B}| = 3 \times 2^4 - 3 = 45$, and $|B| = 3^4 - 3 \times 2^4 + 3 = 36$.