Finals : sample

Part I: Logic

Exercise 1

We want to prove: $((\neg p) \wedge (\neg p \rightarrow q)) \Leftrightarrow (\neg p \wedge q)$

p	q	$a = \neg p$	$b = \neg p \rightarrow q$	$c = a \wedge b$	$a \wedge q$
F	F	T	F	F	F
F	T	T	T	T	T
T	F	F	T	F	F
T	T	F	T	F	F

From columns 5 and 6, we get $((\neg p) \wedge (\neg p \rightarrow q)) \Leftrightarrow (\neg p \wedge q)$.
Using direct derivation we get

$$\begin{aligned}
((\neg p) \wedge (\neg p \rightarrow q)) &\Leftrightarrow (\neg p) \wedge (p \vee q) \\
&\Leftrightarrow (\neg p \wedge p) \vee (\neg p \wedge q) \\
&\Leftrightarrow F \vee (\neg p \wedge q) \\
&\Leftrightarrow (\neg p \wedge q)
\end{aligned}$$

Exercise 2

It is circular reasoning, as the statement depends on its own proposition. The first half that "public transportation is necessary" means "public needs public transportation".

Part II : Proofs & Number Theory

Exercise 1

We disprove that $2^n + 1$ is prime, for all $n \geq 0$:
For $n = 3$, $2^3 + 1 = 9$, which is not prime. Hence, $2^n + 1$ need not be prime, for all $n \geq 0$.

Exercise 2

We want to prove that : $\sqrt[3]{3}$ is irrational.
Suppose $\sqrt[3]{3}$ is not rational. Thus $\sqrt[3]{3} = \frac{p}{q}$, where p and q are coprime integers and q is non-zero.Thus, cubing both sides, we get,$p^3 = 3q^3$, which means 3 divides p^3. Based on Euclid's first proposition (i.e. if a prime number p divides a product ab then p divides a or p divides b), 3 divides p.

Let $p = 3k$, then $27k^3 = 3q^3 \Rightarrow q^3 = 9k^3$. This means that q^3 is a multiple of 3, and using Euclid's first proposition again, we get that q is a multiple of 3. Thus, p and q have a common factor, 3, and this contradicts with our premise that they are coprime. Hence, by contradiction, we have proved that $\sqrt[3]{3}$ is irrational.

Exercise 3

We want to prove : If $9|10^{n-1} - 1$, then $9|10^n - 1$. Since $9|10^{n-1} - 1$, there exists an integer k, such that, $10^{n-1} - 1 = 9k \Rightarrow 10^{n-1} = 9k + 1$. Then, $10^n - 1$ can be written as,

$$\begin{aligned} 10^n - 1 &= 10.10^{n-1} - 1 = 10.(9k + 1) - 1 \\ &= 90k + 10 - 1 = 90k + 9 = 9(10k + 1) \end{aligned}$$

Since 9 is a factor of $10^n - 1$, this proves that if $9|10^{n-1} - 1$, then $9|10^n - 1$.

Exercise 4

We want to prove : $n^2 - n + 5$ is odd for all integers n.
We follow a direct proof. Let n be an integer. If $n = 1$, $n^2 - n + 5 = 5$, which is odd. Now let us suppose $n > 1$. Note that $n^2 - n = n(n - 1)$. n and $n - 1$ are two consecutive integers: one of them is even, and therefore $n(n - 1)$ is even. The sum of an even number and an odd number is odd, therefore $n^2 - n + 5$ is odd for all integers n.

Part III : Proof by Induction

Exercise 1

We want to Prove : $P(n)$ is true, for all $n \geq 1$, where
$P(n)$: $\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$
Let us define LHS(n) $= \sum_{i=1}^{n} i2^i$, and RHS(n) $= (n-1)2^{n+1} + 2$.

- *Basis step:* We want to prove $P(1)$ is true.
  LHS(1) $= 1.2^1 = 2$,
  and
  RHS(1) $= (1 - 1).2^{1+1} + 2 = 0 + 2 = 2$.
  Therefore, LHS(1) = RHS(1): $P(1)$ is true.

- *Inductive step:* Let $P(k)$ be true for an integer $k \geq 1$, which means $\text{LHS}(k) = \text{RHS}(k)$. To prove that $P(k+1)$ is true, we prove that $\text{LHS}(k+1) = \text{RHS}(k+1)$. Let us compute $\text{LHS}(k+1)$:

$$\begin{aligned}
\text{LHS}(k+1) &= \sum_{i=1}^{k+1} i2^i \\
&= \sum_{i=1}^{k} i2^i + (k+1)2^{k+1} \\
&= (k-1)2^{k+1} + 2 + (k+1)2^{k+1} \\
&= 2^{k+1}(k-1+k+1) + 2 \\
&= 2^{k+1}(2k) + 2 \\
&= 2^{k+2}k + 2
\end{aligned}$$

  and

$$\begin{aligned}
\text{RHS}(k+1) &= 2^{(k+1)+1}((k+1)-1) + 2 \\
&= 2^{k+2}k + 2
\end{aligned}$$

  Therefore $\text{LHS}(k+1) = \text{RHS}(k+1)$, i.e. $P(k+1)$ is true.

According to the principle of mathematical induction, we can conclude that $\sum_{i=1}^{n} i2^i = (n-1)2^{n+1} + 2$ for all $n \geq 1$.

Exercise 2

$$\begin{aligned}
a_1 &= 1 \\
a_2 &= a_1 + (2*2-1) = 1+3 = 4 \\
a_3 &= a_2 + (2*3-1) = 4+5 = 9 \\
a_4 &= a_3 + (2*4-1) = 9+7 = 16 \\
a_5 &= a_4 + (2*5-1) = 16+9 = 25
\end{aligned}$$

it seems that $a_k = k^2$.
We will prove that this is true using induction.
Let us define $P(n)$: $a_n = n^2$. We want to prove $P(n)$ is true for all $n \geq 1$.
We already proved the basis case above for $k = 1$ and $k = 2$.
Inductive step: let us suppose $P(k)$ is true for $k \geq 1$. We want to prove

$P(k+1)$ is true.

$$\begin{aligned} a_{k+1} &= a_k + 2(k+1) - 1 \\ &= k^2 + 2k + 2 - 1 \\ &= k^2 + 2k + 1 \\ &= (k+1)^2. \end{aligned}$$

Therefore, $P(k+1)$ is true. According to the principle of mathematical induction, we can conclude that $P(n)$ is true for all $n \geq 1$.

Exercise 3

Let P(m) be $3^{4m} \equiv 1(\text{mod } 10)$

- *Basis case:* Let us prove that $P(1)$ is true,:
  $3^4 = 81 \equiv 1(\text{mod } 10)$. Hence P(1) is true.

- *Inductive step:* Let us assume that $P(n)$ is true. This means that $3^{4n} \equiv 1(\text{mod } 10)$. We want to prove $P(n+1)$.
  Notice that $3^{4(n+1)} = 3^{4n} * 3^4$.
  From the properties of congruence, we know that if $a \equiv b(\text{mod } 10)$ and $c \equiv d(\text{mod } 10)$, then $ac \equiv bd(\text{mod } 10)$. In our case $3^{4n} \equiv 1(\text{mod } 10)$ (premise), and $3^4 \equiv 1(\text{mod } 10)$ (basis step), therefore $3^{4n+4} \equiv 1(\text{mod } 10)$, and therefore $P(n+1)$ is true.

According to the principle of mathematical induction, we can conclude that $P(n)$ is true for all $n \geq 1$.

Exercise 4

Note that in the proof, the basis step states P(1) is true, but does not give details!. However,

$$\sum_{i=1}^{1} i = 1$$

and

$$\frac{(2*1+1)^2}{8} = \frac{3^2}{8} = \frac{9}{8}$$

In fact, $P(1)$ is not true. The basis step is not true and the proof is wrong.

Part IV : Pigeonhole Principle

Exercise 1

Two numbers have their difference divisible by 2006 if they have the same remainder upon division by 2006. There are 2006 possible remainders for the division by 2006: $0, 1, \ldots, 2005$. Let us build a set S of 2007 powers of 3: $S = \{3^0, 3^1, \ldots, 3^{2006}\}$. Let the elements of S be the "'objects"', and the remainders upon division by 2006 the "'boxes"'. There are 2007 objects, that are distributed into 2006 boxes: according to the pigeonhole principle, there is (at least) one box that contains (at least) two objects. These two "'objects"' are two powers of 3 that have the same remainder upon division by 2006, and therefore their difference is divisible by 2006.

Exercise 2

Let us use the hint given to us. Let $S_m = a_1 + a_2 + \ldots + a_m$. There are n such numbers. Let us divide all S_m by n:
$S_m = q_m n + r_m$ with $0 \leq r_m < n$.
There are two cases:

- At least one of the r_m is 0. Then the corresponding S_m is (are) divisible by n. We can set $k = 0$ and $l = m$, and the sum $a_{k+1} + \ldots + a_l = S_m$ is divisible by n.

- None of the r_m are equal to 0. This means that the remainders of the division of the n numbers S_m belongs to $S = \{1, \ldots, n-1\}$. If we define the S_m as "'objects"', and the remainders as "'boxes"', there are n objects, and $n-1$ boxes. According to the Pigeonhole Principle, when the n objects are arranged in the $n-1$ boxes, (at least) one of the boxes contains (at least) two elements. Let r be this box, and S_p and S_q be these two "'objects"'. Then S_p and S_q have the same remainder upon division by n, therefore their difference is divisible by n. Let us suppose $q > p$, then $S_q - S_p = a_{p+1} + \ldots + a_q$. We can set $k = p$ and $l = q$, and the sum $a_{k+1} + \ldots + a_l = S_m$ is divisible by n.

In all cases, we could find k and l such that $a_{k+1} + \ldots + a_l = S_m$ is divisible by n.

Exercise 3

a) Using the complement rule:
   The number of 8-character passwords that can be formed without any

digit $= 26^8$.
The total number of 8-character passwords is 36^8.
Therefore, the number of passwords with at least one digit is $36^8 - 26^8$.

b) To find the number of passwords that contain at least one digit and one letter, we can use the complement and find the number of passwords that do not contain any letter or that do not contain any digit.
From a), we know that the number of passwords that do not contain any digit is 26^8.
Similarly, the number of passwords that do not contain any letter is 10^8.
Since the sets of passwords that do not contain letters and do not contain digits are disjoint, using the sum rule, we can say that the number of passwrods that do not contain any letter or any digit is $26^8 + 10^8$. Thus the number of passwords that contain at least one digit or at least one letter is $36^8 - (26^8 + 10^8)$.