Midterm: Sample 3

ECS20 (Fall 2017)

Part I: logic

1) Using truth tables, establish for each of the two propositions below if it is a tautology, a contradiction or neither.

1) $[p \wedge (q \wedge r)] \rightarrow [((r \wedge p) \wedge q) \vee q]$

   Let us define $A = [p \wedge (q \wedge r)]$ and $B = [((r \wedge p) \wedge q) \vee q]$

p	q	r	$q \wedge r$	A	$r \wedge p$	$(r \wedge p) \vee q$ B	$A \rightarrow B$	
T	T	T	T	T	T	T	T	T
T	T	F	F	F	F	F	T	T
T	F	T	F	F	T	F	F	T
T	F	F	F	F	F	F	F	T
F	T	T	T	F	F	F	T	T
F	T	F	F	F	F	F	T	T
F	F	T	F	F	F	F	F	T
F	F	F	F	F	F	F	F	T

   The proposition $A \rightarrow B$ is a tautology.

2) $[p \leftrightarrow (\neg q \wedge \neg r)] \rightarrow [(\neg(q \wedge r) \rightarrow p]$

   Let us define $A = [p \leftrightarrow (\neg q \wedge \neg r)]$ and $B = [(\neg(q \wedge r) \rightarrow p]$

p	q	r	$\neg q$	$\neg r$	$\neg q \wedge \neg r$	A	$q \wedge r$	$\neg(q \wedge r)$	B	$A \rightarrow B$
T	T	T	F	F	F	F	T	F	T	T
T	T	F	F	T	F	F	F	T	T	T
T	F	T	T	F	F	F	F	T	T	T
T	F	F	T	T	T	T	F	T	T	T
F	T	T	F	F	T	T	T	F	T	T
F	T	F	F	T	F	T	F	T	F	F
F	F	T	T	F	F	T	F	T	F	F
F	F	F	T	T	T	F	F	T	F	T

   The proposition $A \rightarrow B$ is not a tautology and is not a contradiction

3) $[p \leftrightarrow (\neg q \neg r)] \rightarrow [(\neg(q \vee r) \rightarrow p]$

p	q	r	$\neg q$	$\neg r$	$\neg q \wedge \neg r$	A	$q \vee r$	$\neg(q \vee r)$	B	$A \to B$
T	T	T	F	F	F	F	T	F	T	T
T	T	F	F	T	F	F	T	F	T	T
T	F	T	T	F	F	F	T	F	T	T
T	F	F	T	T	T	T	F	T	T	T
F	T	T	F	F	T	T	T	F	T	T
F	T	F	F	T	F	T	T	F	T	T
F	F	T	T	F	F	T	T	F	T	T
F	F	F	T	T	T	F	F	T	F	T

Let us define $A = [p \leftrightarrow (\neg q \wedge \neg r)]$ and $B = [(\neg(q \vee r) \to p]$

The proposition $A \to B$ is a tautology.

2) Smullyan's island

A very special island is inhabited only by Knights and Knaves. Knights always tell the truth, while Knaves always lie. You meet three inhabitants: Alex, John and Sally. Alex says, "John is a Knight if and only if Sally is a Knave". John says, "If Sally is a Knight, then Alex is a Knight". Can you find what Alex, John, and Sally are? Explain your answer.

Let us build the table for the possible options for Alex, John, and Sally. We then check the validity of the two statements, and finally check the consistency of the truth values for those statements with the nature of Alex and John.

Line	Alex	John	Sally	Alex says	John says	Compatibility
1	Knight	Knight	Knight	F	T	No: Alex would be a Knight who lies
2	Knight	Knight	Knave	T	T	Yes
3	Knight	Knave	Knight	T	T	No, John would be a Knave who tells the truth
4	Knight	Knave	Knave	F	T	No, John would be a Knave who tells the truth
5	Knave	Knight	Knight	F	F	No, John would be a Knight who lies
6	Knave	Knight	Knave	T	T	No, Alex would be a Knave who tells the truth
7	Knave	Knave	Knight	T	F	No, Alex would be a Knave who tells the truth
8	Knave	Knave	Knave	F	T	No, John would be a Knave who tells the truth

Therefore Alex and John are Knights and Sally is a Knave.

Part II: proofs

Exercise 1

Use a constructive proof to show that there exist two distinct strictly positive integers whose sum and difference are both perfect squares

This is an existence proof: we just need to find one example! Let $a = 4$ and $b = 5$; then $a + b = 9$, which is a perfect square, and $b - a = 1$, which is a perfect square.

The pair (5,4) is one solution to the problem. There might be others.

Exercise 2

Let a and b be two integers. Show that if either ab or $a+b$ is odd, then either a or b is odd

This is an implication of the form $p \to q$, with:

p: ab is odd or $a+b$ is odd

q: a is odd or b is odd

where a and b are integers.

We use an indirect proof (proof by contrapositive).
Hypothesis: $\neg q$: a is even and b is even.
There exist two integers k and l such that $a = 2k$ and $b = 2l$. Then
$ab = 2k \times 2l = 4kl = 2(2kl)$ therefore there exists an integer $m(= 2kl)$ such that $ab = 2m$: ab is even.
and
$a+b = 2k+2l = 2(k+l)$ therefore there exists an integer $n(= k+l)$ such that $a+b = 2n$: $a+b$ is even.
We have proved that ab is even and $a+b$ is even; $\neg p$ is true. Therefore $\neg q \to \neg p$ is true, and by contrapositive, $p \to q$ is true.

Exercise 3

Let a and b be two integers. Show that if $a^2(b^2-2b)$ is odd, then a is odd and b is odd.

This is an implication of the form $p \to q$, with:

p: $a^2(b^2-2b)$ is odd

q: a is odd and b is odd

where a and b are integers.
We use an indirect proof (proof by contrapositive).
Hypothesis: $\neg q$: a is even or b is even. We look at both cases:

Case 1: a is even.

There exits an integer k such that $a = 2k$. Then $a^2(b^2-2b) = 4k^2(b^2-2b) = 2[2k^2(b^2-2b)]$. Since $2k^2(b^2-2b)$ is an integer, we conclude that $a^2(b^2-2b)$ is even.

Case 2: b is even.

There exits an integer l such that $b = 2l$. Then $a^2(b^2-2b) = a^2(4l^2-4l) = 2[a^2(2l^2-2l)]$. Since $a^2(2l^2-2l)$ is an integer, we conclude that $a^2(b^2-2b)$ is even.

In both cases we have shown that $a^2(b^2-2b)$ is even, i.e. that $\not p$ is true. Therefore $\neg q \to \neg p$ is true, and by contrapositive, $p \to q$ is true.

Exercise 4

Use a proof by contradiction to show that if $(a, b) \in \mathbb{Z}^2$, then $a^2 - 4b \neq 2$.
Let p be the proposition: if $(a, b) \in \mathbb{Z}^2$, then $a^2 - 4b \neq 2$. p is an implication of the form $q \to r$, where $q : (a, b) \in \mathcal{Z}^2$ and $r : a^2 - 4b \neq 2$. We use a proof by contradiction, i.e. we suppose that p is false, which is logically equivalent to q is true and r is false, namely that there exists two integers a and b such that $a^2 - 4b = 2$.
If this is true, then $a^2 = 2 + 4b = 2(1 + 2b)$, i.e. a^2 is even. We know from class that then a is even. As a is even, there exists an integer k such that $a = 2k$. Replacing in the equation,

$4k^2 - 4b = 2$

After division by 2:

$2(k^2 - b) = 1$.

This would mean however that the even number $2(k^2 - b)$ is equal to the odd number 1. We have reached a contradiction. Therefore the hypothesis $\neg p$ is false, is false; we can conclude that p is true.

Extra credit

A very special island is inhabited only by Knights, Knaves, and Normals. Knights always tell the truth, knaves always lie, while Normals may tell the truth or lie. You meet three inhabitants: Alex, Bill, and Corinne. You know that at least one of them is a Knight, and at least one of them is a Knave. Alex says, “Bill is a Knave if and only if Corinne is a Knave”. Bill says, “If Alex is a Knave, then Corinne is a Knave”. Corinne claims, “Alex is a Knave or Bill is a Knight”. Can you find out what type of inhabitant Alex, Bill, and Corinne are? Explain your work.
Let us build the table for the possible options for Alex, Bill, and Corinne. We then check the validity of the three statements, and finally check the consistency of the truth values for those statements with the nature of Alex, Bill, and Corinne.
As at least one of them is a Knight, and at least one of them is a Knave, we can have:

a) Two are Knights, then the last one needs to be a Knave (3 options)

b) Two are Knaves, then the last one needs to be a Knight (3 options)

c) One is a Knight, one is a Knave, and the last one is a Normal (6 options).

We call A=Alex, B=Bill, and C = Corinne.

Therefore Alex is a Knave, Bill is a Knave, and Corinne is a Knight.

Line	A	B	C	A says	B says	C says	Compatibility
1	Knight	Knight	Knave	F	T	T	No: Alex would be a Knight that lies
2	Knight	Knave	Knight	F	T	F	No: Alex would be a Knight that lies
3	Knave	Knight	Knight	T	F	T	No: Alex would be a Knave that tells the trut
4	Knave	Knave	Knight	F	F	T	Possible
5	Knave	Knight	Knave	F	T	T	No:Corinne would be a Knave that tells the tru
6	Knight	Knave	Knave	T	T	F	No:Bill would be a Knave that tells the truth
7	Knight	Knave	Normal	F	T	F	No: Alex would be a Knight that lies
8	Knight	Normal	Knave	F	T	F	No: Alex would be a Knight that lies
9	Knave	Knight	Normal	T	F	T	No: Bill would be a Knight that lies
10	Knave	Normal	Knight	T	F	T	No: Alex would be a Knave that tells the trut
11	Normal	Knight	Knave	F	T	T	No: Corinne would be a Knave that tells the tru
12	Normal	Knave	Knight	F	T	F	No:Bill would be a Knave that tells the truth