Midterm2: Solutions

ECS20 (Fall 2014)

November 20, 2014

Part I: logic

Using truth tables, establish for each of the two propositions below if it is a tautology, a contradiction or neither.

1) $\neg(\neg(p \rightarrow q)) \leftrightarrow (p \leftrightarrow q)$

Notice first that $\neg(\neg(p \rightarrow q))$ is logically equivalent to $p \rightarrow q$.

p	q	$p \rightarrow q$	$p \leftrightarrow q$	$(p \rightarrow q) \leftrightarrow (p \leftrightarrow q)$
T	T	T	T	T
T	F	F	F	T
F	T	T	F	F
F	F	T	T	T

The proposition is neither a tautology nor a contradiction.

c) $P = [p \leftrightarrow (\neg q \wedge \neg r)] \rightarrow [(\neg(q \wedge r)) \rightarrow p]$

p	q	r	$\neg q$	$\neg r$	$\neg q \wedge \neg r$	$p \leftrightarrow (\neg q \wedge \neg r)$	$q \wedge r$	$\neg(q \wedge r)$	$(\neg(q \wedge r)) \rightarrow p$	P
T	T	T	F	F	F	F	T	F	T	T
T	T	F	F	T	F	F	F	T	T	T
T	F	T	T	F	F	F	F	T	T	T
T	F	F	T	T	T	T	F	T	T	T
F	T	T	F	F	F	T	T	F	T	T
F	T	F	F	T	F	T	F	T	F	F
F	F	T	T	F	F	T	F	T	F	F
F	F	F	T	T	T	F	F	T	F	T

The proposition is neither a tautology nor a contradiction.

Part II: logic- puzzles

Exercise 1

A very special island is inhabited only by knights and knaves. Knights always tell the truth, and knaves always lie. You meet three inhabitants: Alex, John and Sally. You ask Alex the question,

”Are you a knight or a knave?” Alex answers, but rather indistinctly, so you cannot make out what he said. You ask then John, ”What did Alex say?” John replies, ”Alex said that he is a knave.” At this point Sally, says, ”Don’t believe John; he is lying!” The question is, what are John and Sally? What about Alex?

The easiest here is to use a table: K means knight, N means knave, T means that the person said the truth, and F that the person said a lie.

Line	Alex	John	Sally	John says	Sally says
1	Knight	Knight	Knight	F	F
2	Knight	Knight	Knave	F	F
3	Knight	Knave	Knight	F	T
4	Knight	Knave	Knave	F	T
5	Knave	Knight	Knight	F	F
6	Knave	Knight	Knave	F	F
7	Knave	Knave	Knight	F	T
8	Knave	Knave	Knave	F	T

Now we note that:

- line 1 cannot be correct: John and Sally would be knights that lie.
- line 2 cannot be correct: John would be a knight that lies.
- line 4 cannot be correct: Sally would be a knave that tells the truth.
- line 5 cannot be correct: John and Sally would be knights that lie.
- line 6 cannot be correct: John would be a knight that lies.
- line 8 cannot be correct: Sally would be a knave that tells the truth.

Lines 3 and 7 are the only possibility, i.e. John is a knave, Sally is a Knight, but we do not know if Alex is a Knight or a Knave.

Exercise 2

Who robbed the National Bank? Inspector Malone knows that the culprit is one and only one of the following: Alex, John, or Sally. He interrogates them. Each makes two statements. Alex: It wasnt me. John did it. John: Listen, Alex did it. And Sally did it. Sally: I didnt do it. Neither did Alex. Each has made one true statement and one false statement. Who did it?

The easiest here is to use a table: G means guilty, N means not guilty, T means that the statement is true, and F that the statement is false. We call PA1 and PA2 the two statements of Alex, PJ1 and PJ2 the two statements of John, and PS1 and PS2 the two statements of Sally.

Line	Alex	John	Sally	PA1	PA2	PJ1	PJ2	PS1	PS2
1	G	N	N	F	F	T	F	T	F
2	N	G	N	T	T	F	F	T	T
3	N	N	G	T	F	F	T	F	T

Now we note that:

- line 1 cannot be correct: Both statements from Alex are true.
- line 2 cannot be correct: Both statements from Alex are true.

Line 3 is the only possibility and it is correct. Sally is guilty.

Part III: proofs

Exercise 1

Prove or disprove that $\forall n > 1$, n integer, there are no integers x, y, and z such that $x^n + y^n = z^n$.

Let n be an integer greater than 1. If we choose $n = 2$, we can find three integers x, y, and z such that $x^2 + y^2 = z^2$. For example, $x = 3$, $y = 4$, and $z = 5$. The property is therefore not true for all $n > 1$.

Exercise 2

Let A and B be two sets in a universe U. Show that $(A \bigcap B) \bigcup \overline{(A \bigcup \overline{B})} = B$.

First solution: using set identities:

$(A \bigcap B) \bigcup \overline{(A \bigcup \overline{B})}$	$=$	$(A \bigcap B) \bigcup (\overline{A} \bigcap \overline{\overline{B}})$	deMorgan's law
	$=$	$(A \bigcap B) \bigcup (\overline{A} \bigcap B)$	double complement
	$=$	$(B \bigcap A) \bigcup (B \bigcap \overline{A})$	commutativity
	$=$	$B \bigcap (A \bigcup \overline{A})$	distributivity
	$=$	$B \bigcap U$	complement law 3
	$=$	B	absorption law 1

Second possible solution: membership table

A	B	$\overline{B}$	$A \bigcap B$	$A \bigcup \overline{B}$	$\overline{A \bigcup \overline{B}}$	$(A \bigcap B) \bigcup \overline{(A \bigcup \overline{B})}$
1	1	0	1	1	0	1
1	0	1	0	1	0	0
0	1	0	0	0	1	1
0	0	1	0	1	0	0

The columns for B and for $(A\bigcap B)\bigcup(A\bar{\bigcup}\bar{B})$ have the same membership values: the two sets are equal.

Exercise 3

Let f be the function from $\mathbb{Z}$ to $\mathbb{Z}$ defined as, $\forall n \in \mathbb{Z}$, $f(n) = n^2 + 4n + 1$. Show that $\forall n \in \mathbb{Z}$, if $f(n)$ is odd, then $f(n+1)$ is even.

We use a direct proof.

Let n be an integer such that $f(n)$ is odd. There exists $k \in \mathbb{Z}$ such that $f(n) = 2k+1$, i.e. $n^2 + 4n + 1 = 2k + 1$.

Let us compute $f(n+1)$:

$$\begin{aligned}
f(n+1) &= (n+1)^2 + 4(n+1) + 1 \\
&= n^2 + 2n + 1 + 4n + 4 + 1 \\
&= n^2 + 4n + 1 + 2n + 1 + 4 \\
&= 2k + 1 + 2n + 1 + 4 \\
&= 2(k+n) + 6 \\
&= 2(k+n+3)
\end{aligned}$$

We can conclude that $f(n+1)$ is even. Therefore the property is true.

Exercise 4

Let A, B, and C be three sets in a universe U. Show that if they satisfy simultaneously $A\bigcup B = C$, $(A\bigcup C)\bigcap B = C$, and $(A\bigcap C)\bigcup B = A$, then they are equal, i.e. $A = B = C$.

We show first that $A = B$. We do this by showing that $A \subset B$ and $B \subset A$.

i) Let us show $A \subset B$.

   Let $x \in A$. Since $A \subset AUB$, $x \in A\bigcup B$. According to the premise, $A\bigcup B = C$; therefore $x \in C$. Since $C = (A\bigcup C)\bigcap B$, $x \in (A\bigcup C)\bigcap B$, hence $x \in A\bigcup C$ and $x \in B$. Therefore $x \in B$. We have shown that any element x in A is also in B. Therefore $A \subset B$.

ii) Let us show $B \subset A$.

   Let x be an element of B. Since $B \subset (A\bigcap C)\bigcup B$, $x \in (A\bigcap C)\bigcup B$. Since $(A\bigcap C)\bigcup B = A$(premise), $x \in A$. We have shown that any element x in B is also in A. Therefore $B \subset A$.

We can conclude that $A = B$.

Let us show now that $B = C$. Note that since $A = B$, $A\bigcup B = B$. We know that $A\bigcup B = C$ (premise). We can conclude that $B = C$.

Extra credit

Every inhabitant of the island of Bahava is either a knight who always tells the truth, a knave who always lies, or a spy who sometimes tells the truth and sometimes lies. Knights, knaves, and spies can be men or women. An old respected tradition on the island is that knights only marry knaves and knaves only marry knights. Hence a spy can marry only a spy. This problem is about two married couples, Mr. and Mrs. A, Mr. and Mrs. B. They are interviewed and three of the four people say: Mr. A: Mr. B is a knight. Mrs. A: My husband is right: Mr. B is a knight. Mrs. B: That is right. My husband is indeed a knight. What are each of the four people?

Just like with all Smullyan's problems, we build a truth table. Mr. A can be a knight, a knave, or a spy, in which case Mrs. A is a knave, a knight, or a spy, respectively. Same for Mr. B and Mrs. B. There are therefore 9 possibilities to consider.

Line #	Mr. A	Mrs. A	Mr. B	Mrs. B	Mr. A says	Mrs. A says	Mrs. B says
1	Knight	Knave	Knight	Knave	T	T	T
2	Knight	Knave	Knave	Knight	F	F	F
3	Knight	Knave	Spy	Spy	F	F	F
4	Knave	Knight	Knight	Knave	T	T	T
5	Knave	Knight	Knave	Knight	F	F	F
6	Knave	Knight	Spy	Spy	F	F	F
7	Spy	Spy	Knight	Knave	T	T	T
8	Spy	Spy	Knave	Knight	F	F	F
9	Spy	Spy	Spy	Spy	F	F	F

We can eliminate:

Line 1: Mrs. A would be a knave that tells the truth

Line 2: Mr. A would be a knight that lies

Line 3: Mr. A would be a knight that lies

Line 4: Mr. A would be a knave that tells the truth

Line 5: Mrs. A would be a knight that lies

Line 6: Mrs. A would be a knight that lies

Line 7: Mrs. B would be a knave that tells the truth

Line 8: Mrs. B would be a knight that lies

The only solution is that all four are spies.