Discrete Mathematics

ECS 20 (Winter 2019)

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Homework 1 - For 1/15/2019

Exercise 1

Let A and B be two natural numbers. Follow the proof given below and identify which step(s) is (are) not valid.

Step #	Equation	Justification
1	$A = B$	Assumption
2	$A \times A = B \times A$	Multiply by B on each side
3	$A^2 - B^2 = AB - B^2$	Subtract B^2 on each side
4	$(A - B)(A + B) = (A - B)B$	Factorize
5	$A + B = B$	Simplify: divide by A-B
6	$B + B = B$	Base on step 1, $A = B$, therefore $A + B = B + B$
7	$2B = B$	By definition, $B + B = 2B$
8	$2 = 1$	Simplify: divide by B

There is only one mistake in the proof, in step 5: we cannot divide by $A - B$ as $A = B$, i.e. $A - B = 0$!!

Exercise 2

Prove the following statements:

a) *The sum of any three consecutive even numbers is always a multiple of 6*

Let N be an odd number. There exists an integer number k such that $n = 2k + 1$. The two odd numbers that follows N are $N + 2$ and $N + 4$, which can be rewritten as $2k + 3$ and

$2k+5$. Let S be the sum of these three consecutive odd numbers. Then:

$$\begin{aligned} S &= N+N+2+N+4 \\ &= 2k+1+2k+3+2k+5 \\ &= 6k+9 \\ &= 3(2k+3) \end{aligned}$$

As $2k+3$ is an integer, S is a multiple of 3. As this is true for all values of N, the proposition is always true.

b) *The sum of any four consecutive odd numbers is always a multiple of 8*

Let N be an odd number. There exists an integer number k such that $N = 2k+1$. The three odd numbers that follows N are $N+2$, $N+4$, and $N+4$, which can be rewritten as $2k+3$, $2k+5$ and $2k+7$. Let S be the sum of these four consecutive odd numbers. Then:

$$\begin{aligned} S &= 2k+1+2k+3+2k+5+2k+7 \\ &= 8k+16 \\ &= 8(k+2) \end{aligned}$$

As $k+2$ is an integer, S is a multiple of 8. As this is true for all values of N, the proposition is always true.

c) *Prove that if you add the squares of two consecutive integer numbers and then add one, you always get an even number.*

Let N be an integer number. The number that follows N is $N+1$. Let S be the sum of the squares of these two consecutive numbers. Then:

$$\begin{aligned} S &= N^2+(N+1)^2 \\ &= N^2+N^2+2N+1 \\ &= 2N^2+2N+1 \end{aligned}$$

Therefore,

$$S+1 = 2(N^2+N+1)$$

As (N^2+N+1) is an integer, $S+1$ is a multiple of 2, i.e. an even number. As this is true for all values of N, the proposition is always true.

Exercise 3

Let x be a real number. Solve the equation $5^{2x}-2(5^x)+1=0$.

Solution: Let x be a real number. Let us define $P(x) = 5^{2x}-2(5^x)+1$. We simplify $P(x)$:

$$\begin{aligned} P(x) &= 5^{2x}-2(5^x)+1 \\ &= (5^x)^2-2(5^x)+1 \end{aligned}$$

Let us define $y = 5^x$. Substituting in the equation above, we get:

$$\begin{aligned} P(x) &= y^2 - 2y + 1 \\ &= (y-1)^2 \end{aligned}$$

Solving $P(x) = 0$ is therefore equivalent to solving $(y-1)^2 = 0$, which has only one solution, $y = 1$. Therefore

$$(5^x) = 1$$

Taking the Log of this equation:

$$xLog(5) = 0$$

Therefore $x = 0$.

Substituting back into $P(x)$: $P(0) = 5^0 - 2 \times 5^0 + 1 = 1 - 2 + 1 = 0$.

Exercise 4

Prove the following identities for p, q, m, n, x, and y real numbers:

a) $8(p-q) + 4(p+q) = 2(p+3q) + 10(p-q)$

Let p and q be two real numbers, and let $LHS = 8(p-q) + 4(p+q)$ and $RHS = 2(p+3q) + 10(p-q)$. Then:

$$\begin{aligned} LHS &= 8p - 8q + 4p + 4q \\ &= 12p - 4q \end{aligned}$$

and

$$\begin{aligned} RHS &= 2p + 6q + 10p - 10q \\ &= 12p - 4q \end{aligned}$$

Therefore $LHS = RHS$ for all p and q, and the identity is true.

b) $x(m-n) + y(n+m) = m(x+y) + n(y-x)$

Let x, y, m and n be four real numbers, and let $LHS = x(m-n) + y(n+m)$ and $RHS = m(x+y) + n(y-x)$. Then:

$$LHS = xm - xn + yn + ym$$

and

$$RHS = xm - xn + ym + yn$$

Therefore $LHS =$ RHS for all x, y, n and m, and the identity is true.

c) $(x+3)(x+8)-(x-6)(x-4)=21x$

Let x be a real number and let $LHS=(x+3)(x+8)-(x-6)(x-4)$ and $RHS=21x$. Then:

$$\begin{aligned} LHS &= x^2+8x+3x+24-x^2+4x+6x-24 \\ &= 21x \\ &= RHS \end{aligned}$$

The identity is true for all x.

d) $m^8-1=(m^2-1)(m^2+1)(m^4+1)$

Let m be a real number and let $LHS=m^8-1$ and $RHS=(m^2-1)(m^2+1)(m^4+1$. Then

$$\begin{aligned} LHS &= (m^4)^2-1^2 \\ &= (m^4-1)(m^4+1) \\ &= ((m^2)^2-1)(m^4+1) \\ &= (m^2-1)(m^2+1)(m^4+1) \\ &= RHS \end{aligned}$$

The identity is true for all m.

Extra credit

Prove that if you add the cubes of two consecutive integer numbers and then add one, you always get an even number.

Let N be an integer number. The number that follows N is $N+1$. Let S be the sum of the cubes of these two consecutive numbers. Then:

$$\begin{aligned} S &= N^3+(N+1)^3 \\ &= N^3+N^3+3N^2+3N+1 \\ &= 2N^3+3N(N+1)+1 \end{aligned}$$

Therefore,

$$S+1 = 2(N^3+1)+3N(N+1)$$

Let us prove now that if N is an integer, then $N(N+1)$ is even.

Proof: N is an integer. We look at two cases:

a) If N is even, there exists an integer k such that $N=2k$. Then $N(N+1)=2k(2k+1)$. Since k(2k+1) is an integer, $N(N+1)$ is even.

b) If N is odd, there exists an integer k such that $N=2k+1$. Then $N(N+1)=2(k+1)(2k+1)$. Since (k+1)(2k+1) is an integer, $N(N+1)$ is even.

Therefore $N(N+1)$ is even for all integer numbers N. There exists an integer k such that $N(N+1)=2k$. Then,

$$\begin{aligned} S+1 &= 2(N^3+1)+6k \\ &= 2(N^3+3k+1) \end{aligned}$$

As (N^3+3k+1) is an integer, $S+1$ is a multiple of 2, i.e. an even number. As this is true for all values of N, the proposition is always true.