Homework 4 Solutions

ECS 20 (Winter 2019)

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Exercise 1

Let n be an integer. Give a direct proof, an indirect proof, and a proof by contradiction of the statement "if n is even, then $n + 9$ is odd."

Let n be an integer. Let p be the proposition: "n is even", and q be the proposition "$n + 9$ is odd".

- **Direct proof**: We show directly $p \to q$.
  Hypothesis: p is true, i.e. n is even.

  As n is even, there exists an integer k such that $n = 2k$, Then, $n + 9 = 2k + 9 = 2(k + 4) + 1$. As $k + 4$ is an integer, $n + 9$ is odd, i.e. q is true. We conclude that $p \to q$.

- **Indirect proof**: We show that $\neg q \to \neg p$.
  Hypothesis: $\neg q$ is true, i.e. $n+9$ is even. Then there exists an integer k such that $n+9 = 2k$. Then $n = 2k - 9 = 2(k - 5) + 1$, i.e. n is odd since $k - 5$ is an integer. $\neg p$ is true.

  We have shown that $\neg q \to \neg p$; by contrapositive, we conclude $p \to q$.

- **Contradiction**:
  Hypothesis: We suppose that the proposition $p \to q$ is false, i.e. that p is true and q is false. Since p is true, n is even, therefore there exists an integer k such that $n = 2k$, Then, $n + 9 = 2k + 9 = 2(k + 4) + 1$. Since $k + 4$ is an integer, $n + 9$ is odd; but the hypothesis states $n + 9$ is even: we reach a contradiction. The hypothesis is wrong, therefore the statement $p \to q$ is true.

Exercise 2

Let $A = \{2, 4, 6\}$ and $B = \{3, 6\}$. Define $\mathcal{P}(A \bigcap B)$, $\mathcal{P}(B)$, and $\mathcal{P}(A \bigcup B)$, where $\mathcal{P}(C)$ indicates the power set of the set C.

Since $A = \{2, 4, 6\}$ and $B = \{3, 6\}$, we have:

a) $A \bigcap B = \{6\}$

b) $A \bigcup B = \{2, 3, 4, 6\}$

Therefore,

a) $\mathcal{P}(B) = \{\emptyset, \{3\}, \{6\}, \{3,6\}\}$

b) $\mathcal{P}(A \bigcap B) = \{\emptyset, \{6\}\}$

c) $\mathcal{P}(A \bigcup B) = \{\emptyset, \{2\}, \{3\}, \{4\}, \{6\}, \{2,3\}, \{2,4\}, \{2,6\}, \{3,4\}, \{3,6\}, \{4,6\}, \{2,3,4\}, \{2,4,6\}, \{3,4,6\}, \{2,3,4,6\}\}$

Exercise 3

Let A, and B be two sets in a universe $\mathcal{U}$. Using set identities (no truth table or membership table!), show the following implications:

There are at least two methods to show set identities. I will use both, namely a "direct" proof based on set theory identity and a proof based on logic, using a membership table.

a) $\left(\overline{(A-B)} \bigcap B\right) - \overline{B} = B$

Let us define $LHS = \left(\overline{(A-B)} \bigcap B\right) - \overline{B}$ and $RHS = B$.

- **Method 1: set identity**

$$\begin{aligned} LHS = \; & \left(\overline{(A-B)} \bigcap B\right) - \overline{B} & \\ & \left(\overline{(A \bigcap \overline{B})} \bigcap B\right) \bigcap B & \text{Property of minus} \\ & ((\overline{A} \bigcup B) \bigcap B) \bigcap B & \text{DeMorgan's Law} \\ & (B \bigcap B) & \text{absorption law} \\ & B & \text{absorption law} \\ & RHS & \end{aligned}$$

- **Method 2: Membership table**

A	B	$\overline{B}$	$(A-B)$	$(\overline{A-B})$	$(\overline{A-B} \bigcap B)$	LHS	RHS
1	1	0	0	1	1	1	1
1	0	1	1	0	0	0	0
0	1	0	0	1	1	1	1
0	0	1	0	1	0	0	0

Since column 7 and 8 are equal, the two sets are equal

b) $\overline{B} - (\overline{B} - A) = \overline{B} \bigcap A$

Let $LHS = \overline{B} - (\overline{B} - A)$ and $RHS = \overline{B} \bigcap A$.

- **Method 1: set identity**

$$\begin{aligned} LHS = \; & \overline{B} - (\overline{B} - A) & \\ & \overline{B} \bigcap \overline{(\overline{B} \bigcap \overline{A})} & \text{property of minus} \\ & \overline{B} \bigcap (B \bigcup A) & \text{DeMorgan's law} \\ & (\overline{B} \bigcap B) \bigcup (\overline{B} \bigcup A) & \text{distributivity} \\ & \overline{B} \bigcup A & \text{complement law and identity} \\ & RHS & \end{aligned}$$

- **Method 2: Membership table**

A	B	$\overline{B}$	$\overline{B}-A$	LHS	RHS
1	1	0	0	0	0
1	0	1	0	1	1
0	1	0	0	0	0
0	0	1	1	0	0

Since column 5 and 6 are equal, the two sets are equal

c) $(A-B)-(A-C)=(A\bigcap C)-B$

Let $LHS=(A-B)-(A-C)$ and $RHS=(A\bigcap C)-B$.

- **Method 1: set identity**

$$
\begin{array}{rll}
LHS = & (A-B)-(A-C) & \\
 & (A\bigcap \overline{B})\bigcap \overline{(A\bigcap \overline{C})} & \text{property of minus} \\
 & (A\bigcap \overline{B})\bigcap(\overline{A}\bigcup C) & \text{deMorgan's law} \\
 & (A\bigcap \overline{B}\bigcap \overline{A})\bigcup(A\bigcap \overline{B}\bigcap C) & \text{distributivity} \\
 & \emptyset\bigcup(A\bigcap C\bigcap \overline{B}) & \text{Absorption law} \\
 & (A\bigcap C)-B & \text{property of minus} \\
 & RHS &
\end{array}
$$

- **Method 2: Membership table**

A	B	C	$A-B$	$A-C$	LHS	$(A\bigcap C)$	B	RHS
1	1	1	0	0	0	1	1	0
1	1	0	0	1	0	0	1	0
1	0	1	1	0	1	1	0	1
1	0	0	1	1	0	0	0	0
0	1	1	0	0	0	0	1	0
0	1	0	0	0	0	0	1	0
0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0

Since column 6 and 9 are equal, the two sets are equal

d) $A - B = (A - B) - (B - C)$

Let $LHS = A - B$ and $RHS = (A - C) - (B - C)$.

- **Method 1: set identity**

$$
\begin{array}{lll}
RHS = & (A - C) - (B - C) & \\
& (A \bigcap \overline{B}) \bigcap \overline{(B \bigcap \overline{C})} & \text{property of minus} \\
& (A \bigcap \overline{B}) \bigcap (\overline{B} \bigcup C) & \text{deMorgan's law} \\
& ((A \bigcap \overline{B}) \bigcap \overline{B}) \bigcup ((A \bigcap \overline{B}) \bigcap C) & \text{distributivity} \\
& (A \bigcap \overline{B}) \bigcup (A \bigcap (\overline{B} \bigcap C)) & \text{associativity} \\
& A \bigcap (\overline{B} \bigcup (\overline{B} \bigcap C)) & \text{distributivity} \\
& A \bigcap \overline{B} & \text{absorption law} \\
& (A - B) & \text{property of minus} \\
& LHS &
\end{array}
$$

- **Method 2: Membership table**

A	B	C	$A-B$	$B-C$	LHS	RHS
1	1	1	0	0	0	0
1	1	0	0	1	0	0
1	0	1	1	0	1	1
1	0	0	1	0	1	1
0	1	1	0	0	0	0
0	1	0	0	1	0	0
0	0	1	0	0	0	0
0	0	0	0	0	0	0

Since column 6 and 7 are equal, the two sets are equal

Exercise 4

Let A, and B be two sets in a universe $\mathcal{U}$. Using set identities (no truth table or membership table!), show the following implications:

a) if $A \bigcup B = B$, then $\overline{(\overline{B} - A)} \bigcap (B \cup A) = B$

We use a direct proof. Let p be the proposition $A \bigcup B = B$ and let q be the proposition $\overline{(\overline{B} - A)} \bigcap (B \cup A) = B$. Our hypothesis is that p is true. Then:

$$
\begin{array}{rll}
\overline{(\overline{B} - A)} \cap (B \cup A) = & \overline{(\overline{B} \cap \overline{A})} \cap (B \cup A) & \text{Property of minus} \\
& (B \cup A) \cap (B \cup A) & \text{DeMorgan's Law} \\
& B \cup A & \text{Absorption law} \\
& B & \text{hypothesis}
\end{array}
$$

Therefore q is true. According to the method of direct proof, $p \to q$ is true.

b) if $A \bigcap B = A$, then $A \bigcup B = B$

We use a direct proof. Let p be the proposition $A \bigcap B = A$ and let q be the proposition $A \bigcup B = B$. Our hypothesis is that p is true. Then:

$$\begin{array}{rll} A \cup B = & (A \cap B) \cup B & \text{Hypothesis: } A = A \bigcap B \\ & B \cup (B \cap A) & \text{Associativity} \\ & B & B \cap A \subset B \end{array}$$

Therefore q is true. According to the method of direct proof, $p \to q$ is true.

Exercise 5

Let A, and B be two sets in a universe $\mathcal{U}$. The symmetric difference of A and B, denoted $A \oplus B$, is the set containing those elements in either A or B, but not in both A and B. This definition can be rewritten as: $A \oplus B = (A - B) \bigcup (B - A)$. Show that:

a) $A \oplus B = (\overline{A} \bigcap B) \bigcup (A \bigcap \overline{B})$

I use a membership table: Since column 3 and column 8 are equal, $(A \oplus B) = (\overline{A} \bigcap B) \bigcup (A \bigcap \overline{B})$.

A	B	$A \oplus B$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap B$	$A \cap \overline{B}$	$(\overline{A} \bigcap B) \bigcup (A \bigcap \overline{B})$
1	1	0	0	0	0	0	0
1	0	1	0	1	0	1	1
0	1	1	1	0	1	0	1
0	0	0	1	1	0	0	0

b) $A \oplus B = (A \bigcup B) - (A \bigcap B)$

Again, I use a membership table:

A	B	$A \cup B$	$A \cap B$	$(A \cup B) - (A \cap B)$	$A \oplus B$
1	1	1	1	0	0
1	0	1	0	1	1
0	1	1	0	1	1
0	0	0	0	0	0

Since column 5 and 6 are equal, $A \oplus B = (A \bigcup B) - (A \bigcap B)$.

Exercise 6

Let A, and B be two sets in a universe $\mathcal{U}$. Show that $|\overline{A} \bigcup \overline{B}| = |\mathcal{U}| - |A| - |B| + |A \bigcup B|$.

Let us define $C = A \bigcap B$.
A complement law tells us that $C \bigcap \overline{C} = U$ where U is the domain. Also, $C \bigcap \overline{C} = \emptyset$. Using the inclusion-exclusion principle, we find:

$$|U| = |C| + |\overline{C}|$$

We know that $|C| = |A \cap B| = |A| + |B| - |A \cup B|$and, based on DeMorgan's law, $\overline{C} = \overline{A} \cup \overline{B}$. Replacing in the equation above, after rearrangement, we obtain:

$$|\overline{A} \bigcup \overline{B}| \quad = \quad |U| - |A| - |B| + |A \bigcup B|$$

Exercise 7

a) Let A, B, and C be three sets in a universe $\mathcal{U}$. Show that $|A \bigcup B \bigcup C| = |A| + |B| + |C| - |A \bigcap B| - |A \bigcap C| - |B \bigcap C| + |A \bigcap B \bigcap C|$.

From the inclusion-exclusion principle, we know that:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

Let us consider the three sets A, B and C. We observe that:

$$|A \cup B \cup C| = |(A \cup B) \cup C|$$

Then:

$$\begin{aligned}
|A \cup B \cup C| &= |A \cup B| + |C| - |(A \cup B) \cap C| \\
&= |A| + |B| + |C| - |A \cap B| - |(A \cup B) \cap C| \\
&= |A| + |B| + |C| - |A \cap B| - |(A \cap C) \cup (B \cap C)| \\
&= |A| + |B| + |C| - |A \cap B| - (|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|) \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |(A \cap C) \cap (B \cap C)| \\
&= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|
\end{aligned}$$

b) In a group of students, 60 play soccer, 40 play hockey, and 36 play cricket, 20 play soccer and hockey, 25 play hockey and cricket, 12 play cricket and soccer and 10 play all the three games. Find the total number of students in the group and how many students only played soccer? (Assume that each student in the group plays at least one game)

Let U be the domain, namely the set of all students. Let S be the set of students who play soccer; $|S| = 60$. Let H be the set of students who play hockey, $|H| = 40$. Let C be the set of students who play cricket, $|C| = 36$. We are also told that $|S \bigcap H| = 20$, $|H \bigcap C| = 26$, $|C \bigcap S| = 12$, and $|S \bigcap H \bigcap C| = 10$.

Let C the set of students that play at least one (type) of sport: $C = S \bigcup H \bigcup C$. Based on part b), we find $|C| = 89$. Since all the students in the group play at least one sport, thus total number of students in the group=89.

Let SH be the set of students who play soccer and hockey, SC the set of students who play soccer and cricket, and SS those students who only play soccer. We note that $S = SH \bigcup SC \bigcup SS$. Also, $|S| = 60$, $|SH| = 20$, $|SC| = 12$, $|SC \bigcap SH| = 10$, and $|SH \bigcap SS| = |SC \bigcap SS| = |SH \bigcap SC \bigcap SS| = 0$. Using part a), we get $|SS| = 38$.

Extra Credit

Let A, B, and C be three sets in a universe $\mathcal{U}$. Using set identities (no truth table or membership table!), show the following identity, $(A \cup B \cup C) \cap \overline{\left(A \cap \overline{B} \cap \overline{C}\right)} \cap \overline{C} = B \cap \overline{C}$

Let $LHS = (A \cup B \cup C) \cap \overline{\left(A \cap \overline{B} \cap \overline{C}\right)} \cap \overline{C}$ and $RHS = B \cap \overline{C}$. Then:

$$
\begin{array}{rll}
LHS = & (A \cup B \cup C) \cap \overline{\left(A \cap \overline{B} \cap \overline{C}\right)} \cap \overline{C} & \\
 & (A \cup B \cup C) \cap \left(\overline{A} \cup B \cup C\right) \cap \overline{C} & \text{DeMorgan's law} \\
 & \left((A \cup B \cup C) \cap \left(\overline{A} \cup B \cup C\right)\right) \cap \overline{C} & \text{Associativity} \\
 & \left(\left(A \cap \overline{A}\right) \cup (B \cap C)\right) \cap \overline{C} & \text{Distributivity} \\
 & (B \cup C) \cap \overline{C} & \text{Absorption Law} \\
 & B \cap \overline{C} & \text{Distributivity and Absorption} \\
 & RHS &
\end{array}
$$