Functions

1) Some definitions about functions

Definition 1: Let A and B be sets. A function f from A to B is an assignement of <u>exactly one</u> element of B to each element of A.

A is the <u>domain</u> of f and B is the <u>co-domain</u>.

Definition 2: A function f from A to B is said to be <u>one to one</u>, or <u>injective</u> if and only if $f(x) = f(y)$ implies $x = y$

$$\forall (x, y) \in A \times A, \quad f(x) = f(y) \rightarrow x = y \quad (1)$$

or

$$\forall (x, y) \in A \times A, \quad x \neq y \rightarrow f(x) \neq f(y) \quad (2)$$

Examples: Let us define the two functions f and g from $\mathbb{R}$ to $\mathbb{R}$ as:

$\forall x \in \mathbb{R}, \quad f(x) = 2x + 1$

$\forall x \in \mathbb{R}, \quad g(x) = x^2$ Are f and g one to one?

- Let us show that f is one to one.

Let x_1, x_2 be two real numbers such that $f(x_1) = f(x_2)$

Then
$$2x_1 + 1 = 2x_2 + 1$$
$$2x_1 = 2x_2$$
$$x_1 = x_2$$

Therefore f is one to one.

- Is g one to one?

$g(2) = 4$ and $g(-2) = 4$ and $2 \neq -2$

We have shown there is a counter-example

Therefore g is not one to one.

<u>Definition</u> A function f from A to B is said to be <u>onto</u>, or <u>surjective</u>, if and only if, for every element b of B, there is an element a of A, with $f(a) = b$

$$\forall y \in B, \ \exists a \in A, \ f(a) = y.$$

<u>Example</u>: Show that the function f defined in the example above is onto.

Let y be a real number.

The equation $2x+1=y$ has one solution: $x_1 = \frac{1}{2}(y-1)$

Therefore $f(x_1) = 2x_1 + 1 = y$.

f is <u>onto</u>.

<u>Definition</u>: A function f from A to B is said to be a <u>one to one correspondence</u>, a a <u>bijection</u>, if it is both injective and surjective.

<u>Definition</u> Let f be a bijection from the set A to the set B. The inverse function of f, f^{-1} if the function that assigns to an element b of B the element a of A such that $f(a) = b$.

<u>Example</u>: $f: \mathbb{R} \to \mathbb{R}$, $x \to 2x+1$ is injective and surjective: it is bijective.

$f^{-1}: \mathbb{R} \to \mathbb{R}$ is defined as:

$x \longrightarrow \frac{1}{2}(x-1)$

Composition of two functions:

Definition: Let g be a function from the set A to the set B, and let f be a function from the set B to the set C. The composition of the functions f and g, denoted $f\circ g$, is defined by:

$$f\circ g: \begin{array}{l} A \rightarrow C \\ a \rightarrow f\circ g(a) = f[g(a)] \end{array}$$

Example:

Let $f: \begin{array}{l} \mathbb{R} \rightarrow \mathbb{R} \\ x \rightarrow 2x+1 \end{array}$ $\qquad g: \begin{array}{l} \mathbb{R} \rightarrow \mathbb{R} \\ x \rightarrow x^2 \end{array}$

The function $f\circ g$ is a function from $\mathbb{R}$ to $\mathbb{R}$ such that: $f\circ g(x) = f[g(x)] = f(x^2) = 2x^2+1$ for all x in $\mathbb{R}$.

What about $g\circ f$? It is also a function from $\mathbb{R}$ to $\mathbb{R}$, defined by:

$$g\circ f(x) = g[f(x)] = g(2x+1) = (2x+1)^2 = 4x^2+4x+1$$

Note: For most functions f and g, $f\circ g \neq g\circ f$

2) Functions and cardinality of sets

Property: We say that two sets A and B have the same cardinality if there exists a bijection from A to B.

Example

$A = \mathbb{Z}$ $\quad$ $B = \mathbb{E}$ (even integers)

Let $f : A \to B$
$x \to 2x$

f is injective:
Let $(x,y) \in \mathbb{Z}^2$ such that $f(x) = f(y)$. Then $2x = 2y$, therefore $x = y$.

f is surjective.
Let $y \in \mathbb{E}$; y is even therefore there exists $k \in \mathbb{Z}$ such that $y = 2k \Rightarrow y = f(k)$

Therefore: $\quad |\mathbb{Z}| = |\mathbb{E}|$

Some definition:

- Any set S with cardinality less than of the natural number, $\mathbb{N}$, or $|S| < |\mathbb{N}|$ is said to be a finite set.
- Any set S that has the same cardinality than $\mathbb{N}$, $|S| = |\mathbb{N}|$ is said to be a countably infinite set

- Any set with cardinality greater than $|\mathbb{N}|$, is said to be uncountable. $\mathbb{R}$ is uncountable

Show that $\mathbb{Z}$ is countably infinite.

Let $f : \mathbb{N} \to \mathbb{Z}$:

n	1	2	3	4	5	6	7	8	9 ...
$f(n)$	0	1	-1	2	-2	3	-3	4	-4

f is bijective. Therefore $|\mathbb{Z}| = |\mathbb{N}|$

3) Floor and ceiling functions

1) Floor function: The floor function assigns to the real number x, the largest integer that is less than or equal to x.

$$\lfloor\ \rfloor : \mathbb{R} \longrightarrow \mathbb{Z}$$
$$x \longrightarrow \lfloor x \rfloor \text{ such that } \lfloor x \rfloor \leq x < \lfloor x \rfloor + 1$$

Examples:

$\lfloor n \rfloor = n \quad \forall n \in \mathbb{N}$

$\lfloor 2.1 \rfloor = 2$

$\lfloor -3.5 \rfloor = -4$

3.2 <u>Ceiling function</u> The ceiling function assigns to the real number x the smallest integer that is greater than or equal to x

$$\lceil\ \rceil : \mathbb{R} \longrightarrow \mathbb{Z}$$
$$x \longrightarrow \lceil x \rceil \text{ such that } \lceil x\rceil - 1 < x \leq \lceil x \rceil$$

3.3 <u>Useful properties of the floor and ceiling functions</u>

a) $x-1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x+1$

(b) $\lfloor -x \rfloor = -\lceil x \rceil$

$\lceil -x \rceil = -\lfloor x \rfloor$

(c) $\lfloor x+n \rfloor = \lfloor x \rfloor + n$

$\lceil x+n \rceil = \lceil x \rceil + n$

3.4) <u>Examples</u>

<u>Example 1</u> Prove that $\forall x \in \mathbb{R}$, $\lfloor 2x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$

<u>Proof</u>: Let x be a real number. We set $LHS = \lfloor 2x \rfloor$ and $RHS = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor$. There exists $n \in \mathbb{Z}$ and $\varepsilon \in \mathbb{R}$, $0 \leq \varepsilon < 1$ such that $x = n + \varepsilon$

By definition of the floor function, $\lfloor x \rfloor = n$

ε is called the fractional part of x (8)

what about $2x$?

$$2x = 2n + 2\varepsilon, \quad \text{with } 0 \le 2\varepsilon < 2$$

Proof by case:

i) $0 \le \varepsilon < \frac{1}{2}$

Then $0 \le 2\varepsilon < 1$

Therefore $2n \le 2n + 2\varepsilon < 2n+1$ i.e $\lfloor 2x \rfloor = 2n$

Similarly:

$$0 \le \varepsilon < \frac{1}{2}$$

$$n + \frac{1}{2} \le n + \varepsilon + \frac{1}{2} < n + 1$$

Since $n < n + \frac{1}{2}$ and $x = n + \varepsilon$, we get:

$$n < x + \frac{1}{2} < n + 1$$

Therefore $\lfloor x + \frac{1}{2} \rfloor = n$

We have: $LHS = \lfloor 2x \rfloor = 2n$

$RHS = \lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = n + n = 2n$

The property is true.

ii) $\frac{1}{2} \le \varepsilon < 1$

Then $1 \le 2\varepsilon < 2$

Therefore $2n+1 \le 2n + 2\varepsilon < 2n+2$ and $LHS = \lfloor 2x \rfloor = 2n+1$

Similarly:

$$\frac{1}{2} \le \varepsilon < 1$$

$$n+1 \le n+\varepsilon+\frac{1}{2} < n+\frac{3}{2} < n+2$$

Therefore: $\lfloor x+\frac{1}{2} \rfloor = n+1$

And $RHS = \lfloor x \rfloor + \lfloor x+\frac{1}{2} \rfloor = 2n+1$

The property is true

Conclusion: In all cases, we have $LHS = RHS$

Example 2 : Prove that $\forall x \ge 0, x \in \mathbb{R}$, $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$

Proof: Let x be a positive real number

Let $m = \lfloor \sqrt{x} \rfloor$ and let $k = \lfloor x \rfloor$

By definition of $\lfloor \sqrt{x} \rfloor$:

$$0 \le m \le \sqrt{x} < m+1$$

Therefore $m^2 \le x < (m+1)^2$ (1)

By definition of $\lfloor x \rfloor$:

$$k \le x < k+1 \quad (2)$$

From (2): $k < (m+1)^2$

From (1): $m^2 \le x$ and k is the largest integer smaller than x $\rightarrow$ $m^2 \le k$

Therefore: $m^2 \le k < (m+1)^2$

$$m \le \sqrt{k} < m+1$$

Therefore: $\lfloor \sqrt{k} \rfloor = m$, or $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$

4. Growth of functions

4.1 Definition: Let f and g be two functions from $\mathbb{R}$ or $\mathbb{Z}$ to $\mathbb{R}$. We say that $f(x)$ is $O(g(x))$ (big "O") if there exists two constants C and k such that

$$\forall x > k, \quad |f(x)| \leq C\,|g(x)|$$

In symbols:

$$\exists C \in \mathbb{R}^+, \exists k \in \mathbb{R}, \quad \forall x > k, \ |f(x)| \leq C|g(x)|$$

Example: Show that $f(x) = x^2 + 2x + 1$ is $O(x^2)$ for $x \in \mathbb{R}$.

We note that: if $x > 1$, $\quad x < x^2$

therefore: $\quad 2x < 2x^2$

We also know: $\quad 1 < x^2$

and $\quad x^2 \leq x^2$

Therefore $\quad x^2 + 2x + 1 \leq 4x^2$

We choose $k = 1$ and $C = 4$

therefore $f(x)$ is $O(x^2)$

4.2. Important theorems

Theorem 1: Let $f(x) = a_n x^n + \dots \dots + a_1 x + a_0$ where x and a_i are real numbers. Then

$$f(x) = O(x^n)$$

Proof: Let x be a real number.

If $x > 1$ then $x^i < x^n$ $\quad 0 \leq i < n-1$

$$|f(x)| \leq |a_n x^n + \dots \dots a_0|$$
$$\leq |a_n| x^n + \dots \dots |a_0|$$
$$\leq |a_n| x^n + \dots + |a_1| x^n + |a_0| x^n$$
$$\leq (|a_n| + \dots |a_0|)\, x^n$$

We choose $k = 1$ and $C = |a_n| + \dots |a_0|$

Theorem 2 Let $f: \mathbb{N} \to \mathbb{R}$

$$n \to 1 + \dots n$$

$f(n)$ is $O(n^2)$

Theorem 3 Let $f: \mathbb{N} \longrightarrow \mathbb{R}$

$$n \longrightarrow n!$$

Then $f(n)$ is $O(n^n)$

Theorem 4 Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$. Then $(f_1 + f_2)(x)$ is $O(\max(g_1(x), g_2(x)))$

Theorem 5: Suppose that $f_1(x)$ is $O(g_1(x))$ and $f_2(x)$ is $O(g_2(x))$. Then $f_1 f_2(x)$ is $O(g_1(x)\,g_2(x))$

Examples: Find a big O estimate for:

- $f(x) = 2x^3 + x^2 \log x$

  $2x^3$ is $O(x^3)$

  $\forall x > 1 \quad x^2 \log x < x^3$ therefore $x^2 \log x$ is $O(x^3)$

  Therefore $f(x)$ is $O(x^3)$

- $f(x) = \dfrac{x^4 + x^2 + 1}{x^3 + 1}$

  For $x > 1$, $f(x) = x \dfrac{1 + \frac{1}{x^2} + \frac{1}{x^4}}{1 + \frac{1}{x^3}} \leq 3x$

  Therefore $f(x)$ is $O(x)$

- More generally:

  $f(x) = \dfrac{a_n x^n + \cdots a_0}{b_p x^p + \cdots b_0}$ is $O\left(\left|\dfrac{a_n}{b_p}\right| x^{n-p}\right)$

4.3. Extending Big-O

Definition Let f and g be functions from $\mathbb{Z}$ or $\mathbb{R}$ to $\mathbb{R}$. We say that $f(x)$ is $\Omega(g(x))$ if there exists constants C and k such that $|f(x)| > C|g(x)|$
($f(x)$ is "big Omega" of $g(x)$)

$$\forall x > k, \ \exists C \in \mathbb{R}^+, \ |f(x)| > C|g(x)|$$

Definition: Let f and g be functions from $\mathbb{Z}$ or $\mathbb{R}$ to $\mathbb{R}$. We say that $f(x)$ is $\Theta(g(x))$ if $f(x)$ is $O(g(x))$ and $f(x)$ is $\Omega(g(x))$
($f(x)$ is "big theta" of $g(x)$, or $f(x)$ is of order $g(x)$)

Example: $1 + \cdots n = \frac{n(n+1)}{2} \geq \frac{n^2}{2}$

Therefore $1 + \cdots n$ is $\Omega(n^2)$

We know that $1 + \cdots n$ is $O(n^2)$

Therefore $1 + \cdots n$ is of order n^2