Addendum to Section 33.4

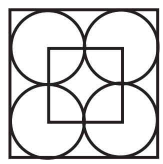
In Section 33.4, we find the following claim.

Claim 1 *A $\delta \times \delta$ box can contain at most four points, when the distance between any two points is at least δ.*

There is no formal proof of this claim. Here, we give a formal proof of a claim which is almost as strong. This kind of argument is called a "packing argument", for reasons which I hope will be obvious.

Claim 2 *A $\delta \times \delta$ box can contain at most five points, when the distance between any two points is at least δ.*

Proof: Consider two points which are at least distance δ from each other. If we put a ball of radius $\delta/2$ around each point, the balls cannot intersect. So another way we can state the claim is that a $\delta \times \delta$ box can contain at most five points, when balls of radius $\delta/2$ around each point do not intersect. This is something like saying we can't pack more than a constant number of oranges of the same size into a constant-sized box, except in this case the oranges can stick out of the box, so long as their centers are inside. To make it exactly like the problem of packing oranges in a box, we just consider a bigger box. Since the centers have to be in the box, no ball can extend outside of the $\delta \times \delta$ box by more than $\delta/2$. We consider enlarging the box by $\delta/2$ in both the x and the y direction. This gives us a larger $2\delta \times 2\delta$ box. The drawing shows a $\delta \times \delta$ box inside of a $2\delta \times 2\delta$ box, with four points in the smaller box, each surrounded by a ball of radius $\delta/2$.



Now the question is, how many balls can be pack into the bigger box? The total area of the box is $4\delta^2$, and the area of each ball is $\pi\delta^2/4$. You can check that $5\pi\delta^2/4 < 4\delta^2$, but $6\pi\delta^2/4 > 4\delta^2$, so we can pack 5 but not six balls into the bigger box.
□

It is certainly sufficient for the analysis of the algorithm to show that there are at most five points in each $\delta \times \delta$ box, but it really seems like four is the right answer. How can you improve this proof to show that there can be at most four?