Minima and Saddles in the MLS Surface Definition

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Shlomo Gortler and Zachary Abel (a Harvard undergrad) point out (personal communication) a problem in the proof of Claim 1 in our paper, "Defining Point-Set Surfaces" [1]. The Claim concerns the MLS surface [2], which is defined in terms of the MLS energy function $e_{\mathrm{MLS}}(y, a)$, a function on the space $\mathbb{R}^3 \times P^2$. We first define the set J_x of point-direction pairs $\{(y, a) \,\|\, a = (y-x)/\mathrm{len}(y-x)\}$, that is, a is the unit direction vector from x to y. Then Levin's definition can be stated:

Definition: A point x belongs to the MLS surface if and only if x is a local minimum of $e_{\mathrm{MLS}}(y, a)$ restricted to the set J_x.

Our Claim was:

Claim 1 *The MLS surface consists of the points for which $n(x)$ is well-defined, and for which*

$$x \in arglocalmin_{y \in \mathcal{L}_{x,n(x)}} e_{MLS}(y, n(x)) \tag{1}$$

Here x is a point in $\mathbb{R}^3$, $n(x)$ is the unit direction vector minimizing $e_{\mathrm{MLS}}(x, a)$ over all a, and $\mathcal{L}_{x,n(x)}$ is the line through x with direction $n(x)$.

This Claim is an if-and-only-if statement; unfortunately only one direction is true. A point of the MLS surface does indeed satisfy Equation (1), but as Shlomo and Zachary point out there may be points which satisfy (1) which are *not* points of the MLS surface. We argued that since x is a minimum along three independent curves in the three-dimensional space J_x, then it had to be a minimum; but this is not necessarily the case. The following one-way version of the Claim (omitting the word "the") does still hold:

Claim 2 *The MLS surface consists of points for which $n(x)$ is well-defined, and for which*

$$x \in arglocalmin_{y \in \mathcal{L}_{x,n(x)}} e_{MLS}(y, n(x))$$

We replace the other direction with a somewhat weaker version:

Claim 3 *A point x which has Property 1 is a minimum or a saddle point of $e_{MLS}(y, a)$ restricted to the set J_x.*

This is weaker in that the MLS surface is defined to include only minima.

Proof: The d-dimensional set J_x can be naturally parameterized by a, the direction vector, and the distance t such that $y = x + ta$. Since $n(x)$ is defined to be the direction that minimizes $e_{\mathrm{MLS}}(x, a)$ over all a, two components of the gradient of e_{MLS} (the dimensions corresponding to a) restricted to J_x are zero. The remaining component, corresponding to t, is also zero, by the definition of the property. Since the gradient at $(x, n(x))$ is zero, $(x, n(x))$ is a critical point within J_x. Since it is a minimum along many directions, it cannot be a maximum, so it has to be either a saddle or a minimum. ∎

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1 Discussion

The main point of the paper, that the MLS surface is a subset of an implicit surface, is not affected by this change. The essential idea is that the MLS surface is a subset of an extremal surface, and every extremal surface is a subset of an implicit surface. The distinctions between the MLS surface and the extremal surface, and the extremal surface and the implicit surface, involve the choice of which critical points over J_x are included: the MLS surface includes only minima, the extremal surface includes some saddles, the implicit surface includes yet more saddles.

In practice, the distinction between the extremal surface and the MLS surface seems negligible. I find it difficult to construct a point set which produces points of the extremal surface which are not also minima of J_x.

References

[1] N. Amenta and Yong Joo Kil, Defining point-set surfaces. *SIGGRAPH* 2004, pages 264-270.

[2] D. Levin, Mesh-independent surface interpolation. In *Geometric Modeling for Scientific Visualization*, G. Brunnett, B. Hamann, K. Mueller, and L. Linsen, Eds. Springer-Verlag.