V. Greedy Algorithms

Greedy algorithms – Overview

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- Greedy algorithms do not always yield optimal solutions,

$$\text{Local optimum} \quad \stackrel{?}{\Longrightarrow} \quad \text{Global optimum}$$

  but *for many problems they do*.

Activity-selection problem

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- Without loss of generality, assume

$$f_1 \leq f_2 \leq \cdots \leq f_n$$

Activity-selection problem

Example

i	s_i	f_i
1	1	4
2	3	5
3	0	6
4	5	7
5	3	8
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$A = \{1, 4, 8, 11\}$ is an optimal (why?) solution.
$A = \{2, 4, 9, 11\}$ is also an optimal solution.

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- Intuitively, this choice leaves as much opportunity as possible for the remaining activities to be scheduled
- That is, the greedy choice is the one that maximizes the amount of unscheduled time remaining.

Activity-selection problem

```
Greedy_Activity_Selector(s,f)
n = length(s)
A = {1}
j = 1
for i = 2 to n
    if s[i] >= f[j]
       A = A U {i}
       j = i
    end if
end for
return A
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Remarks

- Assume the array `f` already sorted
- Complexity: $T(n) = O(n)$

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Solution $A = \{1, 4, 8, 11\}$ by `Greedy_Activity_Selector`.

Activity-selection problem

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Answer: Yes!

Theorem. Algorithm Greedy_Activity_Selector produces a solution of the activity-selection problem.

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  3. *A' is also optimal, since $|A'| = |A|$*

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  1. *the sets $A - \{k_1\}$ and $\{1\}$ are disjoint*
  2. *the activities in A' are compatible*
  3. *A' is also optimal, since $|A'| = |A|$*
- *Therefore, we conclude that* there always exists an optimal solution that begins with a greedy choice.

Activity-selection problem

Property 2.

If A is an optimal solution, then $A' = A - \{1\}$ is an optimal solution to $S' = \{i \in S, s[i] \geq f[1]\}$.

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Proof: By contradiction. If there exists B' to S' such that $|B'| > |A'|$, then let

$$B = B' \cup \{1\},$$

we have

$$|B| > |A|,$$

which is contradicting to the optimality of A.

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Proof of **Theorem**: By Properties 1 and 2, we know that

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Proof of **Theorem**: By Properties 1 and 2, we know that

- After each greedy choice is made, we are left with an optimization problem of the same form as the original.
- *By induction* on the number of choices made, making the greedy choice at every step procedures an optimal solution.

Therefore, the `Greedy_Activity_Selector` produces an optimal solution of the activity-selection problem.

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- Property 2 is called **the optimal substructure property**, generally casted as

  an optimal solution to the problem contains within it optimal solution to subprograms.

These are two key properties for the success of greedy algorithms!