II. Growth of Functions and Asymptotic Notations

Overview

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- Study a way to describe the growth of functions in the limit – *asymptotic efficiency*

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- Focus on what's important (leading factor) by abstracting lower-order terms and constant factors
- Indicate running times of algorithms
- A way to compare "sizes" of functions

$$O \approx \leq$$
$$\Omega \approx \geq$$
$$\Theta \approx =$$

In addition,

$$o \approx <$$
$$\omega \approx >$$

O-notation

- Definition. $g(n)$ is an asymptotic upper bound for $f(n)$, denoted by

$$f(n) = O(g(n))$$

  if there exist constants c and n_0 such that

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$$2n + 10 \leq n^2 \quad \text{for } n \geq 5,$$
  it is true for $c = 1$ and $n_0 = 5$.

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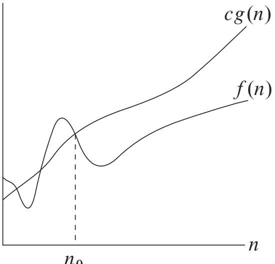

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  Alternative proof: Observe that
$$2n + 10 \leq 2n^2 + 10n^2 = 12n^2 \quad \text{for } n \geq 1,$$
  it is true for $c = 12$ and $n_0 = 1$.

More on O-notation

- $O(g(n))$ is a **set** of functions

$$O(g(n)) = \{f(n) : \ \exists \ c, n_0 \ \text{ s.t. } \ 0 \leq f(n) \leq cg(n) \ \text{ for } n \geq n_0\}$$

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- Examples of functions in $O(n^2)$:
  - $n^2 + n$
  - $n^2 + 1000n$
  - $1000n^2 + 1000n$
  - $n/1000$
  - $n^2/\lg n$

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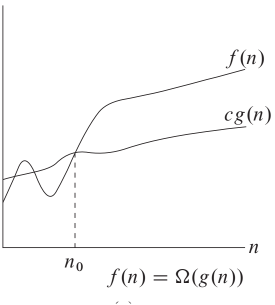

- Example
  - $\sqrt{n} = \Omega(\lg n)$ by picking $c = 1$ and $n_0 = 16$

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- $\Omega(g(n))$ is a **set** of functions

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- Examples of functions in $\Omega(n^2)$:
  - n^2
  - $n^2 + n$
  - $n^2 - n$
  - $1000n^2 + 1000n$
  - $1000n^2 - 1000n$
  - $n^{2.00001}$
  - $n^2 \lg n$
  - n^3

Θ-notation

- Definition. $g(n)$ is an asymptotic tight bound for $f(n)$, denoted by

$$f(n) = \Theta(g(n))$$

if there exist constants c_1, c_2 and n_0 such that

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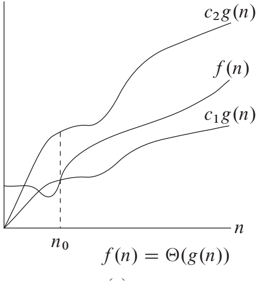
    Since we can pick $c_1 = \frac{1}{4}$ $c_2 = \frac{1}{2}$ and $n_0 = 8$.
  - If $p(n) = \sum_{i=1}^{d} a_i n^i$ and $a_d > 0$, then $p(n) = \Theta(n^d)$



More on Θ-notation

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- Examples of functions in $\Theta(n^2)$:
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Theorem

Theorem. O and Ω iff Θ.

Using limits for comparing orders of growth

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2. $L = \infty$: $f(n) = \Omega(g(n))$
3. $L \neq 0$ is finite: $f(n) = \Theta(g(n))$
4. There is no limit: this technique cannot be used to determine the asymptotic relationship between $f(n)$ and $g(n)$.

Review: L'Hopital's rule

L'Hopital's rule. Let $f(x)$ and $g(x)$ be differential functions with derivatives $f'(x)$ and $g'(x)$, respectively, such that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty.$$

Then

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}.$$

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2. $f(n) = n^{100}$ and $g(n) = 2^n$

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$$n^{100} = O(2^n)$$

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$$n^2 = \Omega(n \lg n)$$

2. $f(n) = n^{100}$ and $g(n) = 2^n$

$$n^{100} = O(2^n)$$

3. $f(n) = 10n(n+1)$ and $g(n) = n^2$

Examples for using limits and L'Hopital's rule for comparing orders of growth

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$$n^{100} = O(2^n)$$

3. $f(n) = 10n(n+1)$ and $g(n) = n^2$

$$10n(n+1) = \Theta(n^2)$$

Reading assiment

Read Section 3.2 of the textbook to review
Standard notations and common functions

1. Monotonicity
2. Floors and ceilings
3. Modular arithmetic
4. Polynomials
5. Exponentials
6. Logarithms
7. Factorials