Problem 4 of Homework 8

Problem statement:

The 3-COLOR problem is known to be NPC. Use this fact to prove that the 4-COLOR is NP-complete.

Proof: By the definition of NPC,

1) We need to show that $4\text{-}\textsc{Color} \in \mathrm{NP}$.

   Given a graph G, and a coloring assignment of vertices, then walk the graph and make certain that all adjacent vertices have a different color and only 4 colors are used. This is done in $O(V + E)$.

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Proof, *cont'd*

2) We need to show that $3\text{-Color} \leq_T 4\text{-Color}$.

Step A: Definition of the transformation T:

Let G^3 be an instance of 3-Color. Construct a new graph G^4 as follows: Add a single extra vertex v and connect it to every other vertex in the graph. This is clearly polynomial in the size of the graph.

Step B: show that

$$\text{Yes of } G^3 \quad \Longleftrightarrow \quad \text{Yes of } G^4$$

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Proof, *cont'd*

- "$\Longrightarrow$" Assume G^3 is 3-colorable. Then G^4 is 4-colorable because the added vertex v, which is connected to all the other vertices in the graph, can be colored with a 4th color, and it will always be connected to vertices that are 1 of 3 other colors.
- "$\Longleftarrow$" Assume G^4 is 4-colorable. Because v is connected to every vertex, v must be the only vertex in G^4 that has a certain color, and all other vertices are colored 1 of 3 other colors. Therefore, G^3 is 3-colorable.

Conclusion: Since we have shown that $4\text{-}\textsc{Color} \in \mathrm{NP}$ and $3\text{-}\textsc{Color} \leq_T 4\text{-}\textsc{Color}$, then we hae shown that $4\text{-}\textsc{Color} \in \mathrm{NP}\text{-}\textsc{Complete}$. $\square$