1. A **recurrence relation** for the sequence $\{a_n\}$ is an equation that expresses a_n in terms of one or more of the previous terms of the sequence, namely, $a_0, a_1, \ldots, a_{n-1}$, for all integers n with $n \geq n_0$, where n_0 is a nonnegative integer.

   The *initial conditions* for a sequence specify the terms that precede the first term where the recurrence relation takes effect.

   Exercise: Let $a_n = a_{n-1} - a_{n-2}$ for $n \geq 2$ with the initial conditions $a_0 = 3$ and $a_1 = 5$. What are the values of a_2, a_3 and a_4?

2. Examples of counting modeled by recurrence relations:

   - Compound interest: $P_n = (1+r)P_{n-1}$ with annual interest rate r, and the initial deposit P_0.
   - Fibonacci numbers $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0, F_1 = 1$.
   - The Tower of Hanoi: $H_n = 2H_{n-1} + 1$

3. A sequence $\{a_n\}$ is called a **solution** of a recurrence relation if its terms satisfy the recurrence relation.

   Example: Let $a_n = 2a_{n-1} - a_{n-2}$ for $n = 2, 3, 4, ...$ Determine whether the following sequences are solutions for every nonnegative integer n:

   - $a_n = 3n$
   - $a_n = 2^n$
   - $a_n = 5$

4. Solving *simple* recurrence relations by direct iterative approach

   Exercises:

   - Find the solution of $a_n = a_{n-1} + 3$ with $a_1 = 2$
   - Find the solution of $a_n = 2a_{n-1} + 1$ with $a_1 = 1$.

5. A **linear kth-order recurrence relation with constant coefficients** is a recurrence relation of the form

   $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k} + f(n), \tag{1}$$

   where $c_1, c_2, \ldots c_k$ are constants (do not depend on n), and $c_k \neq 0$. $f(n)$ is a function of n. If $f(n) = 0$, then the relation is also called to be *homogeneous*. Otherwise, it is called *nonhomogeneous*.

   The recurrence relation

   $$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_k a_{n-k}.$$

   is called the associated homogeneous recurrence relation of (1).

   A consequence of the second principle of mathematical induction is that a sequence satisfying the above recurrence relation is *uniquely* determined by this recurrence relation and the k initial conditions $a_0 = \beta_0$, $a_1 = \beta_1$, $\ldots$, $a_{k-1} = \beta_{k-1}$.

6. A linear **second-order homogeneous recurrence relation** with constant coefficients is a recurrence relation of the form

   $$a_n = s a_{n-1} + t a_{n-2}$$

   where s, t are constants (do not depend on n), and $t \neq 0$.

7. Theorem A-1. For the linear second-order homogeneous recurrence relation, suppose that the characteristic equation

$$r^2 - sr - t = 0$$

has two distinct roots r_1 and r_2. Then the solution of the recurrence relation is of the general form

$$a_n = \alpha r_1^n + \beta r_2^n,$$

where the constants α and β may be uniquely determined using the initial conditions.

Exercises: find the solutions of the following recurrence relations:

- $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$.
- $f_n = f_{n-1} + f_{n-2}$, with the initial conditions $f_0 = 0$ and $f_1 = 1$.

8. Theorem A-2. For the linear second-order homogeneous recurrence relation, suppose that the characteristic equation

$$r^2 - sr - t = 0$$

has only one root r_0 (multiplicity is two). Then the the solution of the recurrence relation is of the general form

$$a_n = \alpha r_0^n + \beta n r_0^n$$

where the constants α and β may be uniquely determined using the initial conditions.

Exercises: find the solutions of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 6$.

9. For a nonhomogeneous recurrence relation, such as

- $a_n = c_1 a_{n-1} + f(n)$,
- $a_n = c_1 a_{n-1} + c_2 a_{n-2} + f(n)$.

Every solution is of the form

$$\{a_n^{(p)} + a_n^{(h)}\},$$

where

- $a_n^{(h)}$ is a solution of the associated homogeneous recurrence relation
- $\{a_n^{(p)}\}$ is a particular solution of the nonhomogeneous recurrence relation

Therefore, the key to solving the nonhomogenous recurrence relations is finding a particular solution.$\{a_n^{(p)}\}$. Although there is no general method for finding such a solution for every function $f(n)$, we can make an educated guess for a certain types of functions $F(n)$. This is illustrated in the following two examples.

Exercises:

(a) Find all solutions of $a_n = 3a_{n-1} + 2n$. What is the solution with the initial condition $a_1 = 3$?

(b) Find all solutions of $a_n = 5a_{n-1} - 6a_{n-2} + 7^n$.