I. Basic rules

1. **The Sum Rule**: If a task can be done *either* in one of n_1 ways *or* in one of n_2 ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $n_1 + n_2$ ways to do the task.

   In set notation: let A_1 be the set of the first n_1 ways to do the task, and A_2 the set of the second n_2 ways to do the task. Assume that A_1 and A_2 are *disjoint*, i.e., $A_1 \cap A_2 = \emptyset$, then the set of ways to the task is $A_1 \cup A_2$ and the number of ways is

$$n(A_1 \cup A_2) = n(A_1) + n(A_2) = n_1 + n_2.$$

2. The extended sum rule: if $A_i \cap A_j = \emptyset$ whenever $i \neq j$, then

$$n(A_1 \cup A_2 \cup \cdots \cup A_m) = n(A_1) + n(A_2) + \cdots + n(A_m) = n_1 + n_2 + \cdots + n_m$$

3. Example 1. How many possible projects are there to choose from the three lists of 23, 15 and 19 projects, respectively?

   Example 2. In how many ways can we get a total of six when rolling two dice?

4. **The Product Rule:** Suppose that a procedure can be broken into a sequence of two tasks. If there are n_1 ways to do the first task, and for each of these ways of doing the first task, there are n_2 ways to do the second task, then there are $n_1 n_2$ ways to do the procedure.

   In set notation: let A_1 be the set of n_1 ways to do the first task, and A_2 the set of n_2 ways to do the second task. Then the set of ways to the procedure is $A_1 \times A_2$ and the number of ways is

$$n(A_1 \times A_2) = n(A_1)n(A_2) = n_1 n_2.$$

5. The extended product rule: $n(A_1 \times A_2 \times \cdots A_m) = n(A_1)n(A_2)\cdots n(A_m) = n_1 n_2 \cdots n_m$.

6. Example 3. How many numbers in the range $1000 - 9999$ do not have any repeated digits?

   Example 4. How many one-to-one functions are there from a set with m elements to a set with n elements?

   Example 5. What is the the number of subsets of a finite set of cardinality n.

   Example 6. How many 10-digits telephone numbers of the format $NXX - NXX - XXXX$, where $N \in \{2, 3, \ldots, 9\}$ and $X \in \{0, 1, 2, \ldots, 9\}$?

7. Many counting problems cannot be solved using just the sum rule or just the product rule, but can be solved using both of these rules in combination.

   Example 7. Each user on a computer system has password, which is six to eight characters long, where each character is an lower letter or a digit. Each password must contain at least one digit. How many poissible passwords are there?

8. **The Inclusion-Exclusion Rule:** Suppose that a task can be done in n_1 or n_2 ways, but that some of the set of n_1 ways to do the task are the same as some of the n_2 ways to do the task. Then first add n_1 and n_2 ways, and afterward subtract the number of ways to do the task that is both among the set of n_1 ways and the set of n_2 ways.

   In set notation.

$$n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2)$$

   Example 8. how many bit strings of length eight start with a "1" bit or end with the two bits "00"?

II. Mathematical functions

1. Factorial function:

$$f(n) = n! = n \cdot (n-1) \cdots 2 \cdot 1$$

   convention: $0! = 1$.

2. Binomial coefficient function:

$$f(n,r) = \frac{n!}{r!(n-r)!}$$

   notation: write $f(n,r)$ as $C(n,r)$ or $\left(\begin{array}{c} n \\ r \end{array}\right)$

3. The binomial theorem: Let x and y be variables, and let n be a positive integer. Then

$$(x+y)^n = \sum_{j=0}^{n} C(n,j) x^{n-j} y^j.$$

   Proof method 1: use mathematical induction.

   Proof method 2: use a combinatorial proof.

4. Example 9. What is the coefficient of $x^{12}y^{13}$ in the expansion of $(x+y)^{25}$?

   Example 10. Show that $\sum_{j=0}^{n} C(n,j) = 2^n$. [1]

5. Pascal's identity

$$C(n+1,k) = C(n,k-1) + C(n,k)$$

   Proof: use algebraic proof or use a combinatorial proof.

6. Pascal's triangle.

[1] We note that this identity can be used for as a combinatorial proof for the fact that the number of subsets of a finite set of elements n is 2^n.

III. Permuation and Combination

1. An r-**permutation** is an ordered arrangement of r elements out of a set of n distinct objects (i.e., no repetitions).

2. By the product rule, we have

$$\begin{aligned} P(n,r) &= \text{the total number of } r\text{-permutations} \\ &= n(n-1)(n-2)\cdots(n-r+1) \end{aligned}$$

3. Example 11. In how many ways can we select a chair, vice-chair, treasurer, secretary from a group of 10 people?

4. An r-**combination** is an *unordered* selection of r elements from a set of n elements.

5. Let us use $C(n,r)$ to denote the total number of r-combinations of a set with n elements, then by the definitions of permutation and combination, and by the product rule, we have

$$P(n,r) = C(n,r) \cdot P(r,r).$$

   Therefore

$$C(n,r) = \frac{P(n,r)}{P(r,r)} = \frac{n!}{r!(n-r)!}.$$

   This is the binomial coefficient function!

6. Example 12. How many ways can we select a committe of two women and three men from a group of 5 women and 6 men?

IV. The Pigeonhole Principles

1. *The pigeonhole principle*: If $k+1$ or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

2. *The Generalized Pigeonhole Principle*: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

3. Example 13. Among any group of 367 people, there must be at least two people with the same birthday, because there are only 366 possible birthdays.

   Example 14. Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

   Example 15. What is the least number of area codes needed to guarantee that the 25 million phones in a state have distinct 10-digits telephone numbers $NXX - NXX - XXXX$, where $N \in \{2,3,\ldots,9\}$ and $X \in \{0,1,2,\ldots,9\}$?

Answers to Examples

1. $n(A_1) + n(A_2) + n(A_3) = 23 + 15 + 19$.

2. $n(A_1) + n(A_2) + n(A_3) = 1 + 2 + 2$. since $A_1 = \{(3, 3)\}$, $A_2 = \{(1, 5), (5, 1)\}$, $A_3 = \{(2, 4), (4, 2)\}$.

3. $9 \cdot 9 \cdot 8 \cdot 7$.

4. If $m > n$, there is no one-to-one function. Otherwise, there are $n(n-1)\cdots(n-m+1)$ one-to-one functions.

5. 2^n.

6. $8 \cdot 10 \cdot 10 \cdot 8 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10 \cdot 10$.

7. Let P_6 = number of possible passwords of length 6, P_7 = number of possible passwords of length 7, and P_8 = number of possible passwords of length 8. Then the total number of possible passwords is $P = P_6 + P_7 + P_8$. Note that the number of characters = 26, and the number of digits = 10.

   P_6 = number of letters and digits − number of strings no digits $= 36^6 - 26^6$.

   Similarily, $P_7 = 36^7 - 26^7$ and $P_8 = 36^8 - 26^8$.

8. The set of the bit strings of eight start with "1" $= A_1 = \{1xxxxxxx \,|\, x \in \{0, 1\}\}$.

   The set of the bit strings of eight start with "00" $= A_2 = \{xxxxxx00 \,|\, x \in \{0, 1\}\}$.

   The set of the bit strings of eight start with "1" and end with "00"

   $= A_1 \cap A_2 = \{1xxxxx00 \,|\, x \in \{0, 1\}\}$.

   Therefore, $n(A_1 \cup A_2) = n(A_1) + n(A_2) - n(A_1 \cap A_2) = 2^7 + 2^6 - 2^5$.

9. $\begin{pmatrix} 25 \\ 13 \end{pmatrix}$

10. Take $x = y = 1$ in the binomial theorem.

11. $10 \cdot 9 \cdot 8 \cdot 7$.

12. $C(5, 2) \cdot C(6, 3)$

13. $\lceil 366/365 \rceil = 2$

14. $\lceil 100/9 \rceil = 9$

15. There are 8 million different phone numbers of the form $NXX - XXXX$. Hence, among 25 million phones, at least $\lceil 25,000,000/8,000,000 \rceil = 4$ area codes are required to ensure that all 10-digit numbers are distinct.