1. An **experiment** is a procedure that yields one of a given set of possible outcomes.

   The **sample space** of the experiments is the set of all possible outcomes, denoted as S.

   An **event** is a subset of the sample space, denoted as E. $E \subseteq S$.

2. If S is a finite sample space of equally likely outcomes, and E is an event, that is, a subset of S, then the **probability** of E is
$$p(E) = \frac{n(E)}{n(S)}.$$

   Property: $0 \le p(E) \le 1$.

   Example 1. A box contains four blue balls and five red balls. What is the probability that a ball randomly chosen from the box is blue?

   Example 2. What is the probability that when two dice are rolled, the sum of the numbers on two dice is 7?

   Example 3. Lottery awards people who correctly choose a set of six numbers out of the first n positive integers. What is the probability that a person picks the correct six numbers out of 40?

3. **Theorem.** Let E be an event in a sample space S. The probability of the complementary event $\bar{E}$ is given by
$$p(\bar{E}) = 1 - p(E).$$

   This is an alternative strategy for finding the probability of an event when a direct approach does not work well.

   Example 4. A sequence of 10 bits is randomly generated. What's the probability that at least one of these bits is 0?

   Example 5. A box contains six red balls and four black balls. Four balls are selected at random from the box.

   (a) What is the probability the event E that two of the selected balls will be red and two will be black?

   (b) If F is the event that no more than two of the balls are red, compute the probability of F.

   (c) If G is the event that no more than three of the balls are red, compute the probability of G.

4. The probability of combinations of events

   Theorem. Let E_1 and E_2 be events in the sample space S, then
$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2).$$

   Example 6. What is the probability that a positive integer selected at random from the set of positive integers not exceeding 100 is divisible by either 2 or 5?

5. Let E and F be events with $p(F) > 0$. The **conditional probability** of E given F, denoted by $p(E \mid F)$, is defined as

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)}$$

   Example 7. A bit string of length four is generated at random. What is the probability that it contains at least two consecutive 0s, given that its first bit is a 0?

   Example 8. What is the conditional probability that a family with two children have two boys, given they have at least one boy?

6. The events E and F are **independent** if and only if $p(E \cap F) = p(E)p(F)$.

   Example 9. Suppose E is the event that a randomly generated bit string of length four begins with a 1 and F is the event that its bit string contains an even number of 1s. Are E and F independent?

   Example 10. continue Example 8, assume that each of the four ways of a family can have two children is equally likely. Are the event E that a family with two children has two boys, and the event F that a family two children has at least one boy independent?

7. Solutions of Examples

Example 1. There are 9 possible outcomes. Hence the probability of that a blue ball is chosen is 4/9.

Example 2. There are a total of 36 outcomes, $S = \{(i, j) \mid 1 \leq i, j \leq 6\}$ and $n(S) = 36$. The set E of successful outcomes: $E = \{(1, 6), (2, 5), (3, 4), (4, 3), (5, 2), (6, 1)\}$. Therefore, $p(E) = \dfrac{n(E)}{n(S)} = 6/36 = 1/6$.

Example 3. There is only one winning combination. The total number of ways to choose six numbers out of 40 is $C(40, 6) = 40!/(34!6!) = 3,838,380$. Therefore, the probability of picking a winning combination is $1/3,838,380 \approx 0.00000026$.

Example 4. Let E be the event that at least one of the 10 bits is 0. $\bar{E}$ is the event that all bits are 1s. Therefore,

$$p(E) = 1 - p(\bar{E}) = 1 - \frac{n(\bar{E})}{n(S)} = 1 - \frac{1}{2^{10}} = \frac{1023}{1024}.$$

Example 5.

(a).

$$p(E) = \frac{C(6, 2) \cdot C(4, 2)}{C(10, 4)} = \frac{15 \times 6}{210} = \frac{90}{210} \approx 0.42.$$

(b)

$$p(F) = \frac{C(6, 0) \cdot C(4, 4) + C(6, 1) \cdot C(4, 3) + C(6, 2) \cdot C(4, 2)}{C(10, 4)} = \frac{1 + 24 + 90}{210} = \frac{115}{210} \approx 0.55.$$

(c) Method 1: use the same way of solving problem (b), we have

$$\begin{aligned} p(G) &= \frac{C(6, 0) \cdot C(4, 4) + C(6, 1) \cdot C(4, 3) + C(6, 2) \cdot C(4, 2) + C(6, 3) \cdot C(4, 1)}{C(10, 4)} \\ &= \frac{1 + 24 + 90 + 80}{210} = \frac{195}{210} \approx 0.93. \end{aligned}$$

Method 2: Let $\bar{G}$ be the event that all four balls are red. Then

$$p(G) = 1 - p(\bar{G}) = 1 - \frac{C(6, 4)}{C(10, 4)} = 1 - \frac{15}{210} = \frac{195}{210} \approx 0.93.$$

Example 6. Let E_1 be the events that the integer selected is divisible by 2, and Let E_2 be the events that the integer selected is divisible by 5. It is easy to see that $n(E_1) = 50$, $n(E_2) = 20$ and $n(E_1 \cap E_2) = 10$. It follows that

$$p(E_1 \cup E_2) = p(E_1) + p(E_2) - p(E_1 \cap E_2) = \frac{50}{100} + \frac{20}{100} - \frac{10}{100} = \frac{3}{5}.$$

Example 7. Let E be the event that a bit string of length four contains at least two consecutive 0s:

$$E = \{0000,\ 0001,\ 0010,\ 0011,\ 1000,\ 1001,\ 0100,\ 1100\}.$$

Then $n(E) = 8$ and $p(E) = 8/16$.

Let F be the event that the first bit of a bit string of length four is a 0:

$$F = \{0000,\ 0100,\ 0010,\ 0110,\ 0011,\ 0001,\ 0011,\ 0111\}.$$

Then $n(F) = 8$ and $p(F) = 8/16$.

Since $E \cap F = \{0000,\ 0001,\ 0010,\ 0011,\ 0100\}$, $p(E \cap F) = 5/16$. Therefore, the conditional probability

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{5/16}{8/16} = \frac{5}{8}.$$

Example 8. Let E be the event that a family with two children has two boys, and let F be the event that a family with two children has at least one boy. Then

$$S = \{BB, BG, GB, GG\}, \qquad E = \{BB\}, \qquad F = \{BB, BG, GB\}.$$

Then

$$E \cap F = \{BB\}.$$

It follows that the conditional probability

$$p(E \mid F) = \frac{p(E \cap F)}{p(F)} = \frac{1/4}{3/4} = \frac{1}{3}.$$

Example 9. Since

$$E = \{1000,\ 1001,\ 1010,\ 1011,\ 1100,\ 1101,\ 1110,\ 1111\}$$

and

$$F = \{0000,\ 0011,\ 0101,\ 0110,\ 1001,\ 1010,\ 1100,\ 1111\},$$

we have

$$E \cap F = \{1001,\ 1100,\ 1010,\ 1111\}.$$

Therefore,

$$p(E) = p(F) = 8/16 = 1/2.$$

and

$$p(E \cap F) = 4/16 = 1/4.$$

Because

$$p(E \cap F) = 1/4 = (1/2)(1/2) = p(E)p(F),$$

we conclude that E and F are independent.

Example 10. Continue of Example 8. Because $E = \{BB\}$, and $p(F) = 3/4$, and $p(E \cap F) = 1/4$. Since $p(E \cap F) = 1/4 \neq (1/4)(3/4) = p(E)p(F)$, the events E and F are not inpdendent.