How to Make Theoretically Passive Reduced-Order Models Passive in Practice

Zhaojun Bai Peter Feldmann Roland W. Freund

Bell Laboratories, Lucent Technologies
Murray Hill, NJ 07974–0636, USA

Abstract— **This paper demonstrates that, in general, implementations of circuit reduction methods can produce unstable and non-passive models even when such outcomes are theoretically proven to be impossible. The reason for this apparent contradiction is the numeric roundoff inherent in any finite-precision computer implementation. This paper introduces a new variant of the symmetric, multiport, Padé via Lanczos algorithm (SyMPVL) that, even in practice, is guaranteed to produce stable and passive models for all the circuits characterized by pairs of symmetric, positive semidefinite matrices. The algorithm is based by a new band Lanczos process with coupled recurrences. A number of circuit examples are used to illustrate the results.**

Introduction

In recent years, model reduction for extracted RC(L) circuits has become an important part of the VLSI design methodology. Parasitic extraction programs typically produce large lumped RC (or even RLC) circuits as models of the structures that link the various functional blocks (the so-called *interconnect*). The storage and the analysis of the interconnect model data for an entire VLSI chip, which may contain millions of interconnect structures, will surpass the capabilities of even the most powerful computers.

The method of choice for solving the interconnect-data storage and analysis problems is the reduced-order modeling of the interconnect RC(L) circuits. The powerful model-reduction techniques introduced in the last few years (see, e.g., [1], [2], [3], [4], [5], [6]) achieve typical compressions of several orders of magnitude in the quantity of interconnect data needed to perform all analyses of interest, with no practical loss of accuracy. The reduced-order models are a compact form of representing interconnect model data and, moreover, can be used in time-domain simulations as substitutes of the full-blown interconnect circuits. Such time-domain simulations are performed to verify timing correctness and signal integrity of VLSI designs. The original full-blown interconnect circuit is always stable and passive, being composed solely of passive components. It is desirable that the reduced-order models preserve these properties to ensure that the time-domain simulations remain always stable.

A number of recent papers (such as [7], [4], [5]) have emphasized the importance of producing stable and passive reduced-order models and proposed model reduction algorithms that "guarantee" the preservation of stability and passivity for RC circuits [8], [2] or, with certain accuracy compromises, even for RLC circuits [5], [6].

In this paper we demonstrate that theoretical proofs of stability and passivity are insufficient for practical applications. When implemented on a real-life, finite-precision computer, it is necessary to show that the model-reduction algorithm maintains the stability and passivity properties even in the presence of the inherent numerical roundoff.

We illustrate this point, through the SyMPVL algorithm applied to RC circuits. In [2], [9], this algorithm was proven, at least in theory, to produce only stable and passive models. We briefly review the algorithm, and the proofs of its stability and passivity in the following section. We show that a straightforward implementation may occasionally produce unstable reduced-order models. The apparent contradiction is explained by the effects of numeric roundoff. We then introduce a new variant of the algorithm that guarantees the stability and passivity of the results even in the presence of roundoff. Finally, we illustrate the difference between the two variants of SyMPVL on a number of examples.

The SyMPVL Algorithm

The $m \times m$, m-port impedance matrix of an RC circuit has the expression

$$\mathbf{Z}(s) = \mathbf{B}^{\mathrm{T}}(\mathbf{G} + s\mathbf{C})^{-1}\mathbf{B}, \tag{1}$$

where $\mathbf{C}$ and $\mathbf{G}$ are real $N \times N$ matrices that represent capacitor and resistor contributions, respectively. For RC circuits, both matrices are symmetric and positive semidefinite. The $N \times m$ matrix $\mathbf{B}$ defines the m ports.

Let $s_0 \geq 0$ be any real expansion point such that the matrix $\mathbf{G} + s_0\mathbf{C}$ is positive definite, and let $\mathbf{G} + s_0\mathbf{C} = \mathbf{M}\mathbf{M}^{\mathrm{T}}$ be its Cholesky factorization ($\mathbf{M}$ is a lower triangular matrix). The matrix $\mathbf{Z}(s)$ can then be recast as follows:

$$\begin{aligned}\mathbf{Z}(s) &= \mathbf{B}^{\mathrm{T}}\left(\mathbf{G} + s_0\mathbf{C} + (s - s_0)\mathbf{C}\right)^{-1}\mathbf{B} \\ &= \mathbf{B}^{\mathrm{T}}\left(\mathbf{M}\mathbf{M}^{\mathrm{T}} + (s - s_0)\mathbf{C}\right)^{-1}\mathbf{B} \\ &= \mathbf{L}^{\mathrm{T}}\left(\mathbf{I} + (s - s_0)\mathbf{A}\right)^{-1}\mathbf{L}.\end{aligned} \tag{2}$$

Here $\mathbf{A} = \mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-\mathrm{T}}$, and $\mathbf{L} = \mathbf{M}^{-1}\mathbf{B}$. The matrix $\mathbf{A}$ is symmetric and positive semidefinite.

The m-port impedance matrix $\mathbf{Z}(s)$ is a matrix-valued rational expression of order N, the number of circuit nodes, which can be very high. In [1], [2], we showed that a low order matrix-Padé approximation of $\mathbf{Z}(s)$ can capture the input-output behavior of the multi-port in the frequency range of interest with practically no loss of accuracy.

The SyMPVL algorithm computes efficiently and robustly such a matrix-Padé approximation by means of a variant of the symmetric band Lanczos algorithm [10],

[9]. The band Lanczos process is a generalization of the standard Lanczos process [11] to multiple starting vectors. Specifically, let $\mathbf{A}^{\mathrm{T}} = \mathbf{A} \in \mathbb{R}^{N\times N}$ and let

$$\widetilde{\mathbf{V}}_1 := \begin{bmatrix} \tilde{\mathbf{v}}_1, & \tilde{\mathbf{v}}_2, & \cdots, & \tilde{\mathbf{v}}_m \end{bmatrix} \in \mathbb{R}^{N\times m}.$$

be a matrix with m starting vectors. The after n iterations, the band Lanczos process has generated a sequence of so-called *Lanczos vectors*, namely the columns of the matrix

$$\mathbf{V}_n = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}.$$

The Lanczos vectors build a basis of the subspace spanned by the first n linearly independent columns of the Krylov matrix

$$\mathbf{K}(\mathbf{A}, \widetilde{\mathbf{V}}_1) := \begin{bmatrix} \widetilde{\mathbf{V}}_1 & \mathbf{A}\widetilde{\mathbf{V}}_1 & \cdots & \mathbf{A}^{N-1}\widetilde{\mathbf{V}}_1 \end{bmatrix}.$$

This subspace is the so-called *block Krylov subspace.*

The band Lanczos process generates the Lanczos vectors by means of $(2m+1)$-term recurrences that can be summarized compactly in matrix form as follows:

$$\mathbf{A}\mathbf{V}_n = \mathbf{V}_n\mathbf{T}_n + \begin{bmatrix} 0 & \cdots & 0 & \hat{\mathbf{v}}_{n+1} & \cdots & \hat{\mathbf{v}}_{n+m_c} \end{bmatrix}. \quad (3)$$

Here $\mathbf{T}_n$ is a banded symmetric $n \times n$ matrix of bandwidth $2m+1$, and $m_c = m_c(n)$ is an index that keeps track of so-called "deflation". Initially, $m_c = m$ and then m_c is reset to be $m_c - 1$ every time deflation occurs. Deflation means that the algorithm encounters a linearly dependent column of the Krylov matrix $\mathbf{K}(\mathbf{A}, \widetilde{\mathbf{V}}_1)$ that has to be deleted.

The Lanczos vectors are constructed to be orthonormal:

$$\mathbf{V}_n^{\mathrm{T}}\mathbf{V}_n = \mathbf{I}_n. \quad (4)$$

Moreover, the "auxiliary vectors", $\{\hat{\mathbf{v}}_{n+j}\}_{j=1}^{m_c}$, in (3) are orthogonal to the Lanczos vectors:

$$\mathbf{V}_n^{\mathrm{T}} \begin{bmatrix} \hat{\mathbf{v}}_{n+1} & \cdots & \hat{\mathbf{v}}_{n+m_c} \end{bmatrix} = \mathbf{0}. \quad (5)$$

We note that, in general, the auxiliary vectors are not orthonormal among themselves, and they even do not need to be linearly independent. Possibly linear dependence of the auxiliary vectors is detected and corrected by the deflation scheme that is built into the algorithm. From (3) to (5), it follows that

$$\mathbf{T}_n = \mathbf{V}_n^{\mathrm{T}}\mathbf{A}\mathbf{V}_n. \quad (6)$$

In [1], [12], we showed how to obtain an n-th matrix-Padé approximant [13] $\mathbf{Z}_n$ to $\mathbf{Z}$ from the quantities $\mathbf{T}_n$ and $\boldsymbol{\rho}_n$ generated by the band Lanczos process. Here $\boldsymbol{\rho}_n$ is an upper-triangular matrix given by $\boldsymbol{\rho}_n = \mathbf{V}_n^{\mathrm{T}}\mathbf{L}$. The n-th matrix-Padé approximant is then defined as follows:

$$\mathbf{Z}_n(s) = \boldsymbol{\rho}_n^{\mathrm{T}} \left(\mathbf{I}_n + (s - s_0)\mathbf{T}_n\right)^{-1} \boldsymbol{\rho}_n. \quad (7)$$

Stability and Passivity

Since the original RC circuit is stable, all the poles of the impedance matrix $\mathbf{Z}(s)$ are non-positive. By (2), the poles of $\mathbf{Z}(s)$ are given by

$$p_{\mathbf{A}} = s_0 - \frac{1}{\lambda}, \quad \lambda \in \lambda(\mathbf{A}). \quad (8)$$

Here, $\lambda(\mathbf{A})$ denotes the set of all eigenvalues of $\mathbf{A}$. By (7), all poles of $\mathbf{Z}_n(s)$ are of the form

$$p_{\mathbf{T}_n} = s_0 - \frac{1}{\lambda}, \quad \lambda \in \lambda(\mathbf{T}_n). \quad (9)$$

From (6) and since $\mathbf{A}$ is positive semidefinite, it follows from (8) and (9) that

$$\lambda(\mathbf{T}_n) \le \max \lambda(\mathbf{A}) \le 0 \quad \text{and} \quad \max p_{\mathbf{T}_n} \le \max p_{\mathbf{A}} \le 0.$$

Thus, the poles (9) of $\mathbf{Z}_n(s)$ are all non-positive. Moreover, it can be shown that a possible pole $p_{\mathbf{A}} = 0$ of $\mathbf{Z}_n(s)$ is simple. Altogether, this proves that the reduced-order models defined by $\mathbf{Z}_n$ are stable.

The reduced-order models defined by $\mathbf{Z}_n(s)$ are also passive. It is well known (see, e.g., [14], [15]) that the reduced-order model defined by a matrix-valued rational matrix $\mathbf{Z}_n(s)$ is passive if, and only if, the following three conditions are satisfied:

(i) $\mathbf{Z}_n(s)$ has no poles in $\mathbb{C}_+ = \{\, s \in \mathbb{C} \mid \operatorname{Re} s > 0 \,\}$ (the right half of the complex plane);

(ii) $\mathbf{Z}_n(\bar{s}) = \overline{\mathbf{Z}_n(s)}$ for all $s \in \mathbb{C}$;

(iii) $\operatorname{Re}\left(\mathbf{x}^{\mathrm{H}}\mathbf{Z}_n(s)\,\mathbf{x}\right) \ge 0$ for all $\mathbf{x} \in \mathbb{C}^n$ and $s \in \mathbb{C}_+$.

Condition (i) is satisfied in view of the stability of $\mathbf{Z}_n$. Condition (ii) follows immediately from (7) and the fact that $\mathbf{T}_n$ and $\boldsymbol{\rho}_n$ are are real matrices. Finally, we verify condition (iii). Let s be any complex number with $\operatorname{Re} s > 0$. Note that, by (6), the matrix $\mathbf{T}_n$ is symmetric positive semidefinite. Therefore, we have for all $\mathbf{y} \in \mathbb{C}^n$

$$\operatorname{Re}\left(\mathbf{y}^{\mathrm{H}}\left(\mathbf{I} + \bar{s}\,\mathbf{T}_n\right)\mathbf{y}\right) = \|\mathbf{y}\|_2^2 + (\operatorname{Re} s)\mathbf{y}^{\mathrm{H}}\mathbf{T}_n\mathbf{y} \ge 0 \quad (10)$$

For any given $\mathbf{x} \in \mathbb{C}^p$, we set

$$\mathbf{y} = \left(\mathbf{I} + s\,\mathbf{T}_n\right)\boldsymbol{\rho}_n\mathbf{x}. \quad (11)$$

Then $\mathbf{y} \in \mathbb{C}^n$, and inserting (11) into (10) gives (iii). Hence the reduced-order model given by $\mathbf{Z}_n$ is passive.

Unfortunately, the stability and passivity proofs given above do not take into account the finite precision of any actual numerical computation. Due to roundoff the computed matrix $\mathbf{T}_n$ may have a number of small negative eigenvalues that, in fact, translate to very large positive poles. Indeed, when we ran the straightforward implementation of the SyMPVL algorithm on an extracted RC circuit consisting of over 200,000 resistors and capacitors, the resulting matrix $\mathbf{T}_n$ had a number of negative eigenvalues that correspond to unstable poles, see Figure 1. The instability manifests itself in the time-domain simulation of the reduced-order model, shown in Figure 2.

Banded Lanczos Process With Coupled Recurrences

In theory, we know that the resulting matrix, $\mathbf{T}_n$, must be positive semidefinite. Therefore, it has a factorization

$$\mathbf{T}_n = \mathbf{L}_n^{\mathrm{T}}\mathbf{D}_n\mathbf{L}_n,$$

where $\mathbf{L}_n$ is a unit lower triangular matrix and $\mathbf{D}_n$ is a diagonal matrix with nonnegative diagonal elements.

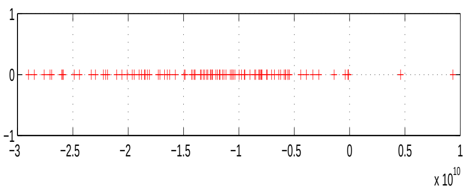

Fig. 1. Reduced-order model poles

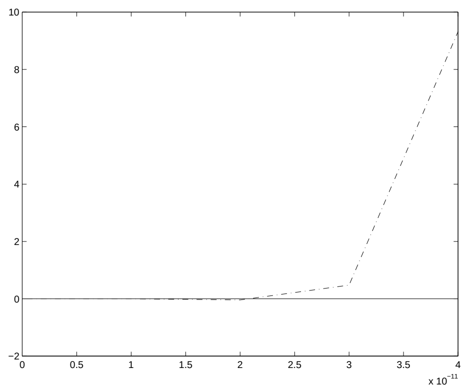

Fig. 2. Time-domain simulation

One way to ensure that, $\mathbf{T}_n$, when computed in finite precision remains positive semidefinite despite the roundoff error is to generate it directly from the factors $\mathbf{L}_n$ and $\mathbf{D}_n$. The proposed new band Lanczos process with coupled recurrences, does exactly that. Instead of generating the entries of $\mathbf{T}_n$, it will compute the elements of the factors $\mathbf{L}_n$ and the diagonal elements of $\mathbf{D}_n$. This way the positive semidefinite property of $\mathbf{T}_n$ is structurally enforced. The algorithm generates two sets of vectors (the columns $\{\mathbf{v}_j\}$ of $\mathbf{V}_n$ and the columns $\{\mathbf{p}_j\}$ of $\mathbf{P}_n$) by means of coupled recurrences that can be summarized as follows:

$$\begin{aligned} \mathbf{A}\mathbf{P}_n &= \mathbf{V}_n\mathbf{L}_n\mathbf{D}_n + \begin{bmatrix} 0 & \cdots & 0 & \widetilde{\mathbf{v}}_{n+1} & \cdots & \widetilde{\mathbf{v}}_{n+m_c} \end{bmatrix}, \\ \mathbf{V}_n &= \mathbf{P}_n\mathbf{L}_n^{\mathrm{T}}. \end{aligned} \tag{12}$$

We refer to $\{\mathbf{p}_j\}$ as the $\mathbf{P}$-vectors. We will see that the above two equations give rise to two coupled recurrences for generating the n-th pair of vectors $\{\mathbf{v}_n\}$ $\{\mathbf{p}_n\}$.

In addition to (12), we also impose the orthogonality (4) and (5), as well as the $\mathbf{A}$-orthogonality conditions

$$\mathbf{P}_n^{\mathrm{T}}\mathbf{A}\mathbf{P}_n = \mathbf{D}_n. \tag{13}$$

From (12), (4), (5), and (13), we have

$$\mathbf{V}_n^{\mathrm{T}}\mathbf{A}\mathbf{V}_n = \mathbf{L}_n^{\mathrm{T}}\mathbf{D}_n\mathbf{L}_n = \mathbf{T}_n. \tag{14}$$

This shows that the band Lanczos process with coupled recurrences directly computes an $\mathrm{LDL}^{\mathrm{T}}$ decomposition of $\mathbf{T}_n$, which can only be positive-semidefinite.

To derive the process to satisfy the equations (12) and (13), we can use recursion from the $(n-1)$-st step to the n-th step. At the n-th step, we need to perform the following steps:

(a) Check for deflation;
(b) Compute the n-th Lanczos vector $\mathbf{v}_n$;
(c) Orthogonalize all auxiliary vectors $\{\mathbf{v}_{n+j}\}_{j=0}^{m_c-1}$ against the new Lanczos vector $\mathbf{v}_n$;
(d) Compute the n-th $\mathbf{P}$-vector $\mathbf{p}_n$;
(e) Construct a new auxiliary vector $\mathbf{v}_{n+m_c}$, which expands the dimension of the underlying block Krylov subspace by one and is orthogonalized against $\mathbf{v}_n$.

Quantitatively, these tasks mean that at the n-th step, we need to compute the vectors $\mathbf{v}_n$, $\mathbf{p}_n$ and the scalars $\{\ell_{nj}\}_{j=n-m_c}^{n-1}$, δ_n. The following algorithm summarizes such a band Lanczos process with coupled recurrences.

Algorithm 1 (Band Lanczos with coupled recurrences)
Input: $\mathbf{A} = \mathbf{A}^{\mathrm{T}} \in \mathbb{R}^{N\times N}$, $\widetilde{\mathbf{V}}_1 = \begin{bmatrix} \widetilde{\mathbf{v}}_1 & \cdots & \widetilde{\mathbf{v}}_m \end{bmatrix} \in \mathbb{R}^{N\times m}$, and the total number of steps n.
Output: $\mathbf{V}_n = \begin{bmatrix} \mathbf{v}_k \end{bmatrix}$, $\mathbf{P}_n = \begin{bmatrix} \mathbf{p}_k \end{bmatrix}$, $\mathbf{L}_n = \begin{bmatrix} \ell_{i,j} \end{bmatrix}$, and $\mathbf{D}_n = \mathrm{diag}(\delta_k)$.

```
m_c = m
for k = 1, ..., n, do
    /* check for deflation */
    η_k = ||ṽ_k||_2
    while η_k = 0
        if m_c ≥ 2,
            for j = 1, 2, ..., m_c − 1,
                ṽ_{k+j−1} = ṽ_{k+j}
            end
        else
            exit /* invariant subspace found */
        end
        m_c := m_c − 1
        η_k = ||ṽ_k||_2
    end
    /* compute k-th Lanczos vector v_k */
    v_k = ṽ_k / η_k
    /* update the existing auxiliary vectors */
    if k − m_c > 0
        ℓ_{k,k−m_c} = η_k / δ_{k−m_c}
    end if
    for j = 1, ..., m_c − 1,
        τ = v_k^T ṽ_{j+1}
        ṽ_{k+j} = ṽ_{k+j} − τ v_k
        if k − m_c + j > 0
            ℓ_{k,k−m_c+j} = τ / δ_{k−m_c+j}
        end if
    end for
    /* compute the k-th vector p_k */
    p_k = v_k
    for j = max(1, k − m_c), ..., k − 1
        p_k := p_k − ℓ_{k,j} p_j
    end for
    /* compute the new auxiliary vector */
    t = A p_k
    δ_k = p_k^T t
    ṽ_{k+m_c} = t − δ_k v_k
end for
```

With our new band Lanczos process (Algorithm 1), the (semi-)positive definiteness of the computed matrix

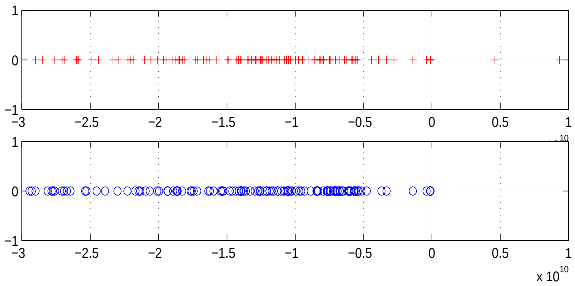

Fig. 3. Ex. 1: dominant poles, SyMPVL(top) and by Alg. 1(bottom)

$\mathbf{L}_n\mathbf{D}_n\mathbf{L}_n^{\mathrm{T}}$ can be detected from the diagonal entries δ_k of $\mathbf{D}_n$. Furthermore, since $\delta_k = \mathbf{p}_k^{\mathrm{T}}\mathbf{A}\mathbf{p}_k$. it is guaranteed that the resulting matrix $\mathbf{L}_n\mathbf{D}_n\mathbf{L}_n^{\mathrm{T}}$ is positive definite if the matrix $\mathbf{A}$ is positive definite, which in turn guarantees the stability and passivity of the reduced order model.

The reduced-order model is obtained by applying Algorithm 1 to matrix $\mathbf{A} = \mathbf{M}^{-1}\mathbf{C}\mathbf{M}^{-\mathrm{T}}$ and starting vectors $\widetilde{\mathbf{V}}_1 = \mathbf{M}^{-1}\mathbf{B}$. After n iteration, we obtain $\mathbf{V}_n$, $\mathbf{L}_n$, and $\mathbf{D}_n$. and $\boldsymbol{\rho}_n = \mathbf{V}_n^{\mathrm{T}}(\mathbf{M}^{-1}\mathbf{B})$. In terms of these quantities, the n-th matrix-Padé approximant $\mathbf{Z}_n(s)$ of $\mathbf{Z}(s)$ is

$$\mathbf{Z}_n(s) = \boldsymbol{\rho}_n^{\mathrm{T}}\left(\mathbf{I}_n + (s - s_0)\mathbf{L}_n\mathbf{D}_n\mathbf{L}_n^{\mathrm{T}}\right)^{-1}\boldsymbol{\rho}_n.$$

By (14) and [1], [12], it is known $\mathbf{Z}_n(s)$ is an nth matrix-Padé approximation of $\mathbf{Z}(s)$, stable and passive.

Examples

The first example is the RC circuit described in the previous section which resulted in an unstable reduced-order model with the classic SyMPVL. The number of Lanczos iterations was 300 due to the large number of ports. On the other hand, SyMPVL with the new coupled recurrences Lanczos process, produced $\mathbf{T}_n = \mathbf{L}_n\mathbf{D}_n\mathbf{L}_n^{\mathrm{T}}$, symmetric positive definite, therefore, all poles are stable, and passivity is preserved. The dominant poles produced by the two algorithms are plotted in Fig. 3.

The second example is another extracted circuit with about 30,000 R and C elements. 60 Lanczos iterations are necessary to account for all the ports. Again, the classic SyMPVL algorithm produced unstable poles, while the new algorithm produced a stable, and passive model. The dominant poles produced by the two algorithms versus the exact ones are plotted in Fig. 4.

Concluding Remarks

This paper has shown that theoretical proofs of stability and passivity of various model-reduction algorithms are in general insufficient. Numerical roundoff, inherent in any practical computer realization of the algorithm, can still cause the loss of these properties, as we illustrated on a couple of realistic circuit examples. Any method that claims guaranteed stability or passivity must be backed by a numerical method capable of translating such a guarantee into practice. This paper has introduced exactly such a method for the SyMPVL algorithm. The newly introduced band Lanczos process with coupled recurrences guarantees that even a finite-precision implementation of SyMPVL produces stable and passive reduced-order models.

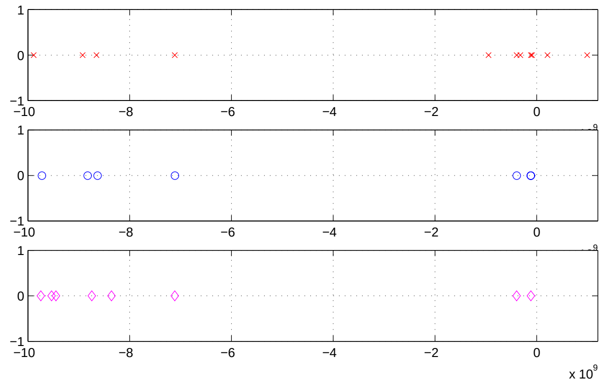

Fig. 4. Ex. 2: SyMPVL (top), Alg. 1 (middle), exact (bottom).

References

[1] P. Feldmann and R.W. Freund, "Reduced-order modeling of large linear subcircuits via a block Lanczos algorithm," in *Proc. 32nd ACM/IEEE Design Automation Conf.*, June 1995.

[2] P. Feldmann and R.W. Freund, "Interconnect-delay computation and signal-integrity verification using the SyMPVL algorithm," in *Proc. 1997 European Conference on Circuit Theory and Design*, Sep. 1997.

[3] R.W. Freund and P. Feldmann, "The SyMPVL algorithm and its applications to interconnect simulation," in *Proceedings of SISPAD'97*, Sep. 1997.

[4] K.J. Kerns and A.T. Yang, "Preservation of passivity during RLC network reduction via split congruence transformations," in *Proc. 34nd ACM/IEEE Design Automation Conf.*, 1997.

[5] A. Odabasioglu, M. Celik, L.T. Pileggi, "PRIMA: passive reduced-order interconnect macromodeling algorithm," in *Tech. Dig. 1997 IEEE/ACM International Conference on Computer-Aided Design*, Nov. 1997.

[6] L.M. Silveira, M. Kamon, I. Elfadel, and J. White, "A coordinate-transformed Arnoldi algorithm for generating guaranteed stable reduced-order models of RLC circuits," in *Tech. Dig. 1996 IEEE/ACM International Conference on Computer-Aided Design*, Nov. 1996.

[7] I.M. Elfadel and D.D. Ling, "Zeros and passivity of Arnoldi-reduced-order models for interconnect networks," in *Proc. 34nd ACM/IEEE Design Automation Conf.*, Jun. 1997.

[8] R.W. Freund and P. Feldmann, "Reduced-order modeling of large passive linear circuits by means of the SyPVL algorithm," in *Tech. Dig. 1996 IEEE/ACM International Conference on Computer-Aided Design*, Nov. 1996.

[9] R.W. Freund and P. Feldmann, "Reduced-order modeling of large linear passive multi-terminal circuits using matrix-Padé approximation," in *Proceedings of DATE'98*, Feb. 1998.

[10] A. Ruhe, "Implementation aspects of band Lanczos algorithms for computation of eigenvalues of large sparse symmetric matrices," *Math. Comp.*, vol. 33, pp. 680–687, 1979.

[11] C. Lanczos, "An iteration method for the solution of the eigenvalue problem of linear differential and integral operators," *J. Res. Nat. Bur. Standards*, vol. 45, pp. 255–282, 1950.

[12] R.W. Freund, "An extension of the Lanczos-Padé connection to the matrix case," Numerical Analysis Manuscript, Bell Laboratories, Murray Hill, NJ, in preparation.

[13] G.A. Baker, Jr. and P. Graves-Morris, *Padé Approximants*, Second Edition, New York, NY: Cambridge University Press, 1996.

[14] M.R. Wohlers, *Lumped and Distributed Passive Networks*, New York, N.Y.: Academic Press, 1969.

[15] B.D.O. Anderson and S. Vongpanitlerd, *Network Analysis and Synthesis*, Englewood Cliffs, NJ: Prentice-Hall, 1973.