Homework Assignment #3

Problem One (1.6.4)

Suppose that A={2, 4, 6}, B={2, 6}, C={4, 6}, and D={4, 6, 8}. Determine which of these sets are subsets of which other of these sets.

Every set is a subset of itself B is a subset of A C is a subset of both A and D.

Problem Two (1.6.14)

What is the cardinality of each of these sets?

a) ∅	0
b) {∅}	1
c) {∅, {∅}}	2
d) {∅, {∅}, {∅, {∅}}}	3

Problem Three (1.6.16)

Can you conclude that A = B if A and B are two sets with the same power set?

Yes. By definition, P(A) is the set of all subsets that can be generated from A, if A and B generate the exact same collection of valid subsets, then it must be that A and B contain the same elements and are therefore equal.

Problem Four (1.6.20)

What is the Cartesian product A cross B, where A is the set of courses offered by the mathematics department at a university and B is the set of mathematics professors at this university?

It is the set of all possible combinations of math courses and possible instructors.

Problem Five (1.7.4)

Let A={a, b, c, d, e} and B={a, b, c, d, e, f, g, h}. Find

a) $A \cup B$	{a, b, c, d, e, f, g, h}
b) $A \cap B$	{a, b, c, d, e}
c) $A - B$	{ }
d) $B - A$	{f, g, h}

Problem Six (1.7.16)

Show that, if A and B are sets, then $(A \cap B) \cup (A \cap B^C) = A$. [**Note**: B^C is another way of writing the complement of set B]

There are two ways of solving set proofs like these, one is to look at an arbitrary point and use the properties of sets to argue why something it true. The other way to do this is to notice that a set is nothing but a collection of elements, and that collection of elements will be the truth set for some proposition. Therefore, if we can prove that the truth sets on either side of the equation are equal, then the sets they represent must also be equal. Both proof forms are provided as a solution. Note that, while the proposition-style proof is inherently longer, it is often far more formulaic in its construction and therefore easier to write and, usually, understand.

<u>Set-style Proof</u>

Note that every point a, in A, is an element of either $A \cap B$ (in the case that $a \in B$) or an element of $A \cap B^C$ (in the case that $a \in B^C$). Also, if $a \in (A \cap B) \cup (A \cap B^C)$, then either $a \in A \cap B$ or $a \in \cap B^C$, in either of these cases, $a \in A$ by the definition of intersection, so it must be that $(A \cap B) \cup (A \cap B^C) = A$.

<u>Proposition-style Proof</u>

Let p(x) be the proposition whose truth set is the set A

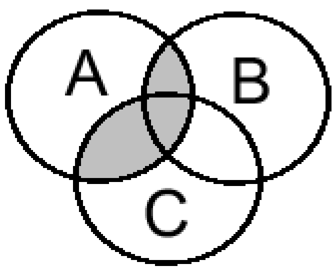
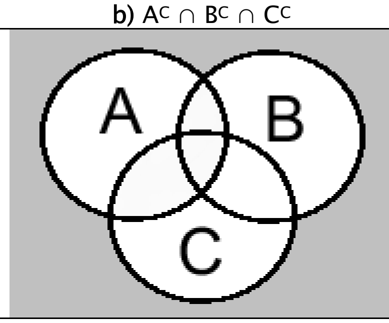
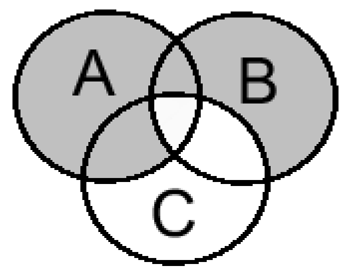
Let q(x) be the proposition whose truth set is B

[01] $A = x \in U, p(x)$	By definition
[02] $B = x \in U, q(x)$	By definition
[03] $(A \cap B) = x \in U, p(x) \wedge q(x)$	By [01] and [02]
[04] $B^C = x \in U, \sim q(x)$	By definition
[05] $(A \cap B^C) = x \in U, p(x) \wedge \sim q(x)$	By [01] and [04]
[06] $(A \cap B) \cup (A \cap B^C) = x \in U, (p(x) \wedge q(x)) \vee (p(x) \wedge \sim q(x))$	By [03] and [05]
[07] $(A \cap B) \cup (A \cap B^C) = x \in U, p(x) \wedge (q(x) \vee \sim q(x))$	By [06] and DeMorgan's Law
[08] $(A \cap B) \cup (A \cap B^C) = x \in U, p(x) \wedge T$	By [07] and the Complement Law
[09] $(A \cap B) \cup (A \cap B^C) = x \in U, p(x)$	By [08] and the Identity Law
[10] $(A \cap B) \cup (A \cap B^C) = A$	By [01]

Problem Seven (1.7.20)

Draw the Venn diagrams for each of these combinations of the sets A, B, C.

We assume for each of these, that the intersection of A, B, and C is non-empty. In each picture, the toned area represents the set in question.

a) $A \cap (B \cup C)$	b) $A^C \cap B^C \cap C^C$	c) $(A - B) \cup (A - C) \cup (B - C)$
		

Problem Eight (1.7.22)

Can you conclude that A=B if A, B, and C are sets such that

a) $A \cup C = B \cup C$? No, this would be true if A and B are both subsets of C.

b) $A \cap C = B \cap C$? No, consider the case when C is the empty set.

Problem Nine (1.7.28)

Show that $A \oplus B = (A - B) \cup (B - A)$.

<u>Set-style Proof</u>

There are only two ways and element, x, can be in set A or set B, exclusive. It can be an element of A but not of B (in which case it is an element of the set A – B) or it can be an element of B but not of A (in which case it is an element of the set B – A). Therefore, an element, x, is in the set $A \oplus B$ only if it is also in the set $(A - B) \cup (B - A)$, so the two sides are equal.

<u>Proposition-style Proof</u>

Let p(x) be the proposition whose truth set is the set A

Let q(x) be the proposition whose truth set is B

Therefore, $(A - B) \cup (B - A) = x \in U, (p(x) \wedge \sim q(x)) \vee (q(x) \wedge \sim p(x))$

[01] $A = x \in U, p(x)$	By definition
[02] $B = x \in U, q(x)$	By definition
[03] $A \oplus B = x \in U, \sim(p(x) \wedge q(x)) \wedge (p(x) \vee q(x))$	By [01] and [02]
[04] $A \oplus B = x \in U, (\sim p(x) \vee \sim q(x)) \wedge (p(x) \vee q(x))$	By [03] and DeMorgan's Law
[05] $A \oplus B = x \in U, [(\sim p(x) \vee \sim q(x)) \wedge p(x)] \vee [(\sim p(x) \vee \sim q(x)) \wedge q(x)]$	By [04]and the Distributive Law
[06] $A \oplus B = x \in U, [F \vee (\sim q(x) \wedge p(x))] \vee [(\sim p(x) \wedge q(x)) \vee F]$	By [05] and the Distributive/Complement Law
[07] $A \oplus B = x \in U, (\sim q(x) \wedge p(x)) \vee (\sim p(x) \wedge q(x))$	By [06] and the Complement Law
[08] $A \oplus B = x \in U, (p(x) \wedge \sim q(x)) \vee (q(x) \wedge \sim p(x))$	By [07] and the Associative Law
[09] $A \oplus B = (A - B) \cup (B - A)$	By [08] and the definition above

Problem Ten (1.7.38)

Let Aj = { ... -2, -1, 0, 1, ..., j}. Find

a) $\bigcup_{j=1}^{n} A_j$

Each A_j is the set { ... j}, so every A_j fully contains the sets A_{j-1} A_{j-2} etc. as subsets. Therefore, the union of the sets A_1 through A_n is exactly A_n. We can take this one step further and say that, since n is unbounded, A_n, in fact, is the set $(-\infty,\infty)$.

b) $\bigcap_{j=1}^{n} A_j$

Each A_j is the set { ... j}, so every A_j fully contains the sets A_{j-1} A_{j-2} etc. as subsets. Therefore, the intersection of the sets A_1 through A_n is exactly A_1 = {... -2, -1, 0, 1}.

Problem Eleven (1.8.2)

Note: the problem listed on the HW#3 handout was the wrong problem >.< Here is the correct problem (solutions to the incorrect problem appear below this one).

Determine whether f is a function from ℤ to ℝ if

a) $f(n) = \mp n$

By definition, a function must map every point in its domain to a single point in the range. Since, for every n, there are two images, +n and -n, this is not a function.

b) $f(n) = \sqrt{(n^2 + 1)}$

Since f(n) maps every point in the domain to some, single point in the range, this is a function.

c) $f(n) = 1/(n^2 - 4)$

By definition, a function must map every point in its domain to some point in the range. Since the points x=2 and x=-2 have no image in the range, this is not a function.

Determine whether f is a function from ℝ to ℝ if

a) $f(x) = 1/x$

By definition, a function must map every point in its domain to some point in the range. Since the point x=0 has no image in the range, this is not a function.

b) $f(x) = \sqrt{(x)}$

By definition, a function must map every point in its domain to some point in the range. Since no value less than 0 has an image in the range, this is not a function.

c) $f(x) = \mp\sqrt{(x^2 + 1)}$

By definition, a function must map every point in its domain to a single point in the range. Since, for every x, there are two images, this is not a function.

Problem Twelve (1.8.12)

Determine whether each of these functions from ℤ to ℤ is one-to-one

a) $f(n) = n - 1$ This function maps each value in ℤ to a unique image, therefore it is one-to-one.

b) $f(n) = n^2 + 1$ The values 2 and -2 map to the same image in ℤ, therefore it is not one-to-one.

c) $f(n) = n^3$ This function maps each value in ℤ to a unique image, therefore it is one-to-one.

d) $f(n) = \lceil n/2 \rceil$ The values 3 and 4 map to the same image in ℤ, therefore it is not one-to-one.

Problem Thirteen (1.8.18)

Determine whether each of these functions is a bijection from ℝ to ℝ

a) $f(x) = -3x + 4$ This function is both one-to-one and onto, therefore it is a bijection.

b) $f(x) = -3x^2 + 7$ This function is neither one-to-one nor onto, therefore it is not a bijection.

c) $f(x) = (x + 1)/(x + 2)$ This is not a function at all since it does not map the value -2.

d) $f(x) = x^5 + 1$ This function is both one-to-one and onto, therefore it is a bijection.

Problem Fourteen (1.8.22)

Let S={-1, 0, 2, 4, 7}. Find f(S) if:

a) $f(x) = 1$ f = {(-1, 1), (0, 1), (2, 1), (4, 1), (7,1)} f(S) = { 1 }

b) $f(x) = 2x + 1$ f = {(-1, -1), (0, 1), (2, 5), (4, 9), (7, 15)} f(S) = {-1, 1, 5, 9, 15}

c) $f(x) = \lceil x/5 \rceil$ f = {(-1, 0), (0, 0), (2, 1), (4, 1), (7, 2)} f(S) = {0, 1, 2}

d) $f(x) = \lfloor (x^2 + 1)/3 \rfloor$ f = {(-1, 0), (0, 0), (2, 1), (4, 5), (7, 16)} f(S) = {0, 1, 5, 16}

Problem Fifteen (1.8.28)

Find fog and gof where $f(x) = x^2 + 1$ and $g(x) = x + 2$ are functions from ℝ to ℝ.

$f o g = f(g(x)) = (x + 2)^2 + 1 = x^2 + 4x + 4 + 1 = x^2 + 4x + 5$

$g o f = g(f(x)) = (x^2 + 1) + 2 = x^2 + 3$

Problem Sixteen (1.8.32)

Let f be a function from the set A to the set B. Let S and T be subsets of A. Show that

a) $f(S \cup T) = f(S) \cup f(T)$

Suppose we have a point b such that f(a) = b for some a ∈ (S ∪ T). Either a∈S and f(a) clearly belongs in f(S), or a∈T and clearly b belongs in f(T), either way, b∈f(S) ∪ f(T). This means that f(S ∪ T) ⊆ f(S) ∪ f(T) is true. Now assume there is a point b such that b∈(f(S) ∪ f(T)). This means that b is either a point in f(S) or b is a point in f(T), this means f has a pre-image (i.e., a point a such that f(a) = b) either in S or in T. In either case, b is an image for some point in (S ∪ T) and is therefore an element of f(S ∪ T). This means that f(S) ∪ f(T ⊆ f(S ∪ T). But if f(S ∪ T) ⊆ f(S) ∪ f(T) and f(S) ∪ f(T ⊆ f(S ∪ T), then it must be that f(S ∪ T) = f(S) ∪ f(T).

b) $f(S \cap T) \subseteq f(S) \cap f(T)$

Suppose we have a point b such that b∈ f(S ∩ T). Then b = f(a) for some a∈(S ∩ T). This implies that a∈S and a∈T, so we know that b∈f(S) and b∈f(T), so b must be an element of f(S) ∩ f(T) and we know that f(S ∩ T) ⊆ f(S) ∩ f(T).