Problem Set 1 Solutions

Problem 1. Let $G = (V, E)$ be a graph (the "usual" sort, being nonempty, finite, undirected, having no-self loops and no multiple edges). Prove (by giving a convincing argument) or disprove (by giving a smallest counter-example) that the following are equivalence relations for any graph G.

Part A. Let $x, y \in V$. Say that $x\ R_G\ y$ if there is a path in G from x to y (that is, a sequence of vertices $x_1, \ldots, x_n \in V$ $(n \geq 1)$ where each $\{x_i, x_{i+1}\} \in E$ and $x = x_1$ and $y = x_n$).

True. $x\ R_G\ x$ by definition; $x\ R_G\ y$ iff $y\ R_G\ x$ because the graph is undirected; and transitivity by concatenating the two paths. of this relation are the "components" of G.) (Note: the equivalence classes of this equivalence relation are called the "components" of G.)

Part B. Let $x, y \in V$. Say that $x\ R_G\ y$ if x is adjacent to y (that is, $\{x, y\} \in E$).

False. The reflexive property fails to hold not just for some graphs, but for all graphs G: $(x, x) \notin R_G$ for any G, because the definition of "adjacent" stated above does not have a vertex being adjacent to itself ("no self-loops"). So any graph G provides a counterexample, and the smallest counter-example is the one-vertex graph.

Part C. Let $x, y \in V$. Say that $x\ R_G\ y$ if $x = y$ or $\{x, y\} \in E$ or there are two vertex-disjoint paths from x to y (paths $x_1, \ldots, x_m$ and $x'_1, \ldots, x'_{n'}$ where $x_1 = x'_1 = x$ and $x_n = x'_{n'} = y$ and $\{x_2, \ldots, x_{n-1} \cap \{x'_2, \ldots, x'_{n'-1} = \emptyset$.

False. The graph G which is a chain of three vertices ($V = \{a, b, c\}$ and $E = \{\{a, b\}, \{b, c\}\}$) violates transitivity, as $(a, b) \in R_G$ and $(b, c) \in R_G$ but $(a, c) \notin R_G$.

Part D. Let $x, y \in V$. Say that $x\ R_G\ y$ if there is a path from x to y and this remains so even if one removes any edge $e \in E$.

True. Clearly $x\ R_G\ x$ and $x\ R_G\ y$ iff $y\ R_G\ x$. Fix $x, y, z \in V$. Suppose that for any edge $e \in E \cup \{\text{extra-edge}\}$ there is a path P_e in G from x to y that doesn't use edge e, and suppose that for any edge $e \in E \cup \{\text{extra-edge}\}$ there is a path P'_e in G from y to z that doesn't use edge e. Then for any edge $e \in E \cup \{\text{extra-edge}\}$ there is a path P''_e in G that doesn't uses e—namely, the concatenation of paths P_e and P'_e. (Note: the equivalence classes of this equivalence relation are called the "strongly-connected components" of G.)

Problem 2. State whether the following propositions are true or false, explaining each answer.

Part A. $\emptyset$ is a language.

True. The empty set is a set of strings (the one with no strings), all over a common alphabet (any alphabet you care to think of).

Part B. $\emptyset$ is a string.

False. A string is a sequence of characters and the empty set is not a sequence of characters.

Part C. ϵ is a language.

False. It is a string, not a language. Of course $\{\epsilon\}$ is a language, but that is quite different from ϵ.

Part D. ϵ is a string.

True. It's the unique string of length zero.

Part E. Every language is infinite or has an infinite complement.

True. If L is infinite we are done, and if L is finite and over some alphabet Σ then $\overline{L} = \Sigma^* - L$ is an infinite set less a finite set, which is finite.

Part F. Some language is infinite and has an infinite complement.

True. Let $L = \{w \in \{0,1\}^* : \ |w| \text{ is odd}\}$.

Part G. The set of real numbers is a language.

False. A language is a set of strings. A real number is not a string (although some real numbers can be represented by strings using a standard encoding).

Part H. There is a language that is a subset of every language.

True. The empty set $\emptyset$ *is such a language.*

Part I. The Kleene-star (Kleene closure) of a language is always infinite.

False. There are two counterexamples: $\emptyset$ *and* $\{\epsilon\}$*, both of which have the finite set* $\{\epsilon\}$ *as their Kleene-star.*

Part J. The concatenation of an infinite language and a finite language is always infinite.

False. But slightly tricky: the only exception is concatenating an infinite language to the emptyset. The concatenation of an infinite language and a nonempty language is indeed infinite.

Part K. There is an infinite language L containing the emptystring and such that L^i is a proper subset of L^* for all $i \geq 0$.

True. An example is $L = \{1^i : \ i \text{ is zero or a power of } 2\}$*. Then* $L^* = \{1\}^*$ *but* L^i *contains only strings whose of length* m *where* m*, when written in binary, contains* i *or fewer 1's.*

Problem 3 Friday is a holiday[1]; don't come to class. Instead, find something to read on the web on César Chavéz. Or join me on Friday from 1:15 pm to 3:15 pm in room 1065 Kemper, when I'll screen the well-regarded documentary *The Fight in the Fields* (1997). But double-check our course web page Thursday night after 11 pm or Friday morning to confirm that I was actually able to get hold of the film.[2]

[1] How annoying is it to have class begin on a Wednesday when the following Friday is a holiday? They couldn't postpone the start of the term until May 2?

[2] Problem 3 will not be graded and the film is an activity unrelated to your work in this particular class.