Rice's Theorem

You may find Rice's theorem helpful in doing some of the problems on the homework or exams and, in general, for understanding part of the reason for the pervasiveness of undecidability. Here is the theorem and its proof.

Definition: $P : \{0,1\}^* \to \{0,1\}$ is a nontrivial property of Turing-acceptable languages if

(1) $P(\langle M \rangle)$ depends only on $L(M)$— if $L(M) = L(M')$ then $P(\langle M \rangle) = P(\langle M' \rangle)$.

(2) P is nontrivial— $\exists M_1$ such that $P(\langle M_1 \rangle) = 1$, and $\exists M_0$ such that $P(\langle M_0 \rangle) = 0$.

If P is a property of Turing-acceptable languages then we say that a language $L = L(M)$ has property P iff $P(\langle M \rangle) = 1$.

Theorem [Rice]: If P is a nontrivial property of Turing-acceptable languages, then

$$L_P = \{\langle M \rangle : \ P(\langle M \rangle) = 1\}$$

is undecidable. (In fact, L_P is not Turing-acceptable if $\emptyset$ has property P; and L_P is not co-Turing-acceptable if $\emptyset$ does not have property P.)

Proof: Case 1. Assume $\emptyset$ does *not* have property P, ie., $P(\langle M_0 \rangle) = 0$, where M_0 is a TM for which $L(M_0) = \emptyset$. We show $A_{\text{TM}} \leq_{\text{m}} L_P$. To show this, we must exhibit a Turing computable function f for which $\langle M' \rangle = f(\langle M, w \rangle)$ is a machine accepting a language with property P iff M accepts w. We specify the behavior of M' on input x to be:

> "Run M on w.
>
> If M rejects, *reject*.
>
> Run M_1 on x, where M_1 is a (fixed) machine for which $P(\langle M_1 \rangle) = 1$. (Here we are using that P is nontrivial).
>
> If M_1 accepts, *accept*; if M_1 rejects, *reject*."

Clearly M' is Turing computable from M and w. Observe that

(1) if M accepts w then $L(M') = L(M_1)$, which is a language with property P.

(2) if M does not accept w, then $L(M') = \emptyset$ which, by assumption, is a language which does not have property P.

Case 2. Assume $\emptyset$ *does* have property P. Proceed as above, reducing $\overline{A_{\text{TM}}}$ to L_P. $\square$