Midterm Solutions

1 Short Answer

(A) *Carefully define what it* ***means*** *if we say:* ***the regular languages are closed under intersection****.*

It means that if L and L' are regular then $L \cap L'$ is regular.

(B) *Carefully state the* ***pumping lemma*** *for* ***regular*** *languages.*

If L is regular then there exists a number p such that for all $s \in L$ where $|s| \geq p$ there is a partitioning of s into $s = xyz$ where $y \neq \varepsilon$ and $|xy| \leq p$ and $xy^iz \in L$ for all $i \geq 0$.

(C) *Draw a DFA that accepts*

$L = \{w \in \{0,1\}^* : w \textit{ is the binary representation of a number divisible by 3}\}$.

Draw it!

(D) *Complete the definition: A context free grammar is a 4-tuple* $\cdots$

$G = (V, \Sigma, R, S)$ where V is a nonempty set and Σ is an alphabet and $V \cap \Sigma = \emptyset$ and R is a finite subset of $V \times (V \cup \Sigma)^*$ and $S \in V$.

(E) *Give a CFG for* $L = \{w \in \{a,b\}^* : w \textit{ is a palindrome}\}$.

$S \to aSa \mid bSb \mid a \mid b \mid \varepsilon$

(F) *Prove that the language of Part (E),* $L = \{w \in \{a,b\}^* : w \textit{ is a palindrome}\}$*, is* ***not*** *regular.*

Suppose for contradiction that L is regular. Let p be the pumping length for L, as guaranteed by the pumping lemma. Let $s = a^pba^p$. By the pumping lemma s can be partitioned into $s = xyz$ where $|xy| \leq p$, $|y| \geq 1$, and $xy^iz \in L$ for all $i \geq 0$. Thus y falls in the original run of a's and xy^0z is a string $z = a^{p'}ba^p$ with $p' < p$, which is not in L. This is a contradiction.

(G) *Using the procedure given in class and in your book, convert the following NFA into a regular expression for the same language.*

Do it!

2 Justified True or False

1. *If F is a finite language and L is an arbitrary language then $L \cap F$ is a regular language.*

   True. The intersection of a finite language and an arbitrary language is finite, and a finite language is regular.

2. *If L^* is regular, then L is regular, too.*

   False. $L = \{1^p : p \text{ is prime}\}$.

3. *Every regular language can be accepted by an DFA $M = (Q, \Sigma, \delta, q_0, F)$ such that $\delta^*(q_0, x) \neq q_0$ for any string $x \in \Sigma^+$.*

   True. Add a new start state and connect it up appropriately.

4. *If L_1 and L_2 are accepted by DFAs then $L_1 \oplus L_2 = (L_1 - L_2) \cup (L_2 - L_1)$ is accepted by a DFA.*

True. Use the product construction.

5. *Suppose L is a regular language with arbitrarily long "gaps" — for every n there is an interval $[a_n, b_n]$ such that $b_n - a_n \geq n$ and L has no strings of any length $\ell \in [a_n .. b_n]$. Then L is finite.*

True. By the pumping lemma, an infinite regular language contains strings of length $a + bi$ for a nonzero constant b and all i.

6. *If a context-free grammar G is ambiguous then it can be converted to a context-free grammar G' where $L(G) = L(G')$ and G' is not ambiguous.*

False. We stated in class that some languages are "inherently ambiguous."

3 A Decision Procedure

Suppose you are given regular expressions α, β over the alphabet $\{a, b\}$. Describe a decision procedure (an algorithm) to answer the following question: is $L(\alpha) \cap L(\beta)$ infinite?

Using the procedures seen in class, convert α and β into NFAs N_α and N_β for $L(\alpha)$ and $L(\beta)$. Using the procedures seen in class, convert N_α and N_β into DFAs M_α and M_β for $L(N_\alpha)$ and $L(N_\beta)$. Using the product construction, convert M_α and M_β into a DFA M for $M_\alpha \oplus M_\beta$. Now use DFS to see if there is a path from the start state s of M to some final state q_f of M where somewhere along this path there is a state q such that there is a nontrivial path from q back to itself. If such a path exists, answer **yes**, otherwise answer **no**.

4 A Closure Property

Suppose that L is a regular language over the alphabet $\Sigma = \{0, 1\}$. Define

$$\mathcal{B}(L) \quad = \quad \{x : xy \in L \text{ for some } y \in \Sigma^*\}$$

Part A. *List the elements of $\mathcal{B}(\{101\})$:*

$\{\varepsilon, 1, 10, 101\}$.

Part B. Prove that if L is regular then $\mathcal{B}(L)$ is regular.

Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA for L. Then a DFA for $\mathcal{B}(L)$ can be constructed as $M' = (Q, \Sigma, \delta, q_0, F')$ where $F' = \{q \in Q : \ \delta^*(q, y) \in F \text{ for some } y \in \Sigma^*\}$.

Question: is it a problem that, above, we finalize even states that are not reachable from q_0?