Practice Final Solutions (Fall 1995 Final)

1 Recall ... [30 points]

A. *Complete the following definition:*

Let $A, B \subseteq \{0,1\}^*$. We say that A polynomial-time mapping reduces to B, written $A \leq_{\mathrm{P}} B$, if ...

there exists a polynomial-time computable function f such that $x \in A$ iff $f(x) \in B$.

B. *Complete the following definition:*

A language L is NP-Complete if ...

(1) $L \in$ NP, and
(2) for all $A \in$ NP, $A \leq_{\mathrm{P}} L$.

C. *State the Cook/Levin Theorem:*

Theorem [Cook/Levin]:

There is an NP-Complete language. In fact, SAT is NP-complete.

D. *Complete the following statement of the pumping lemma for context free languages:*

Theorem. If a language L is context free then there exists a number K such that ...

for all $s \in L$ such that $|s| \geq K$ there exists u, v, x, y, z such that $s = uvxyz$ and $|vxy| \leq K$ and $|vy| \geq 1$ and $uv^ixy^iz \in L$ for all $i \geq 0$.

E. *In a sentence or two, state the "Church-Turing Thesis:"*

Turing machines exactly capture our intuitive notion of what is effectively computable.

F. *State Rice's Theorem (which says that a certain class of languages is undecidable).*

Theorem [Rice]:

If P is a non-trivial property of the r.e. languages then $\{\langle M\rangle : L(M) \text{ has property } P\}$ is undecidable.

2 True/False [20 points]

Mark the correct box by putting an "X" through it. No justification required.

1. For every i, the language $L_i = \{a^ib^ic^i\}$ is context free. **True**
2. Assume L is a regular language and let L_{1101} be the subset of L which contains the strings that end in a 1101. Then L_{1101} is regular. **True**
3. If L^* is decidable then L is decidable. **False**
4. Every enumerable language can be accepted by a TM whose head only moves to the right. **False**

5. For any language L, the language L^* is infinite. **False**

6. Let $\langle M \rangle$ be the encoding of a Turing machine M. Let $P(\langle M \rangle) = 0$ if M on ε halts in an even number of steps, 1 otherwise. Then P satisfies the condition of Rice's theorem: it is a nontrivial property of enumerable languages. **False**

7. The language $L = \{\langle M \rangle : L(M) \in \text{NP}\} \in \text{NP}$. **False**

8. Suppose *3SAT* $\leq_{\text{P}} L$ and $L \in \text{P}$. Then $\text{P} = \text{NP}$. **True**

9. A_{TM} is NP-complete. **False**

10. If $L_1 \leq_{\text{P}} L_2$ and $L_2 \leq_{\text{P}} L_1$, then $L_1 = L_2$. **False**

3 Language Classification. [20 points]

Classify as:

decidable	*decidable*
r.e.	*enumerable but not decidable*
co-r.e.	*co-enumerable but not decidable*
neither	*neither enumerable nor co-enumerable*

No explanation is required.

1. $\{\langle M \rangle : M$ is a TM which accepts some palindrome$\}$. **r.e.**

2. $\{\langle M \rangle : M$ is a TM which accepts some string of length $\geq |\langle M \rangle|\}$. **r.e.**

3. $\{d :$ the digit d appears infinitely often in the decimal expansion of $\pi = 3.14159\cdots\}$. **decidable**

4. $\{\langle G \rangle : G$ is a regular grammar and $L(G)$ contains an even-length string$\}$ **decidable**

5. $\{\langle G \rangle : G$ is a CFG and $L(G) = \Sigma^*\}$ **co-r.e.**

6. $\{\langle G, k \rangle : G$ is a graph containing no clique of size $k\}$. **decidable**

7. A language L for which $L_{\Sigma^*} = \{\langle M \rangle : L(M) = \Sigma^*\} \leq_{\text{m}} L$. **neither**

8. $\{\langle t \rangle : t$ is an instance of the tiling problem for which you can tile the plane using 10 or fewer tile types.$\}$. **Omit this; we did not cover the tiling problem this term.**

4 A Tight Bound on DFA Size [20 points]

Let $L_5 = \{111\}$ be the language over $\{0, 1\}$ which contains only the string 111.

(A) *Show that L_5 can be recognized by an 5-state DFA.*

Just draw the five-state DFA that accepts L_5.

(B) *Prove that L_5 can not be recognized by a 4-state DFA.*

Assume for contradiction that there exists a four state DFA $M = (Q, \Sigma, \delta, q_0, F)$ that accepts L_5. Consider the five states $q_i = \delta^*(q_0, 1^i)$ for $0 \leq i \leq 4$. We claim that all five of these states must be distinct, contradicting the pigeonhole principle. To see this, note that if $q_i = q_I$ for $0 \leq i < I \leq 4$ then $\delta^*(q_0, 1^i) = \delta^*(q_0, 1^I)$ so $\delta^*(q_0, 1^i 1^{3-i}) = \delta^*(q_0, 1^I 1^{3-i})$ so $\delta^*(q_0, 1^3) = \delta^*(q_0, 1^{I+3-i})$ which is absurd, because the first state must be final and the second state must not be.

5 A Decision Procedure [20 points]

If α is a regular expression, we write α^2 for the regular expression $\alpha\alpha$. Show that the following language is decidable (ie., exhibit a decision procedure for this language):

$$L = \{\langle a, b, c\rangle : a,\ b \text{ and } c \text{ are regular expressions and } a^2 \cup b^2 = c^2.\}$$

Using the procedure given in class, convert the regular expression $aa \cup bb$ into an NFA, and then a DFA, M_1. Using the procedure given in class, convert the regular expression cc into an NFA, and then a DFA, M_2. Using the product construction, construct a DFA M the language of which is $L(M_1) \oplus L(M_2)$. Using the procedure given in class, look to see if $L(M)$ is empty. If it is, answer *yes*; otherwise, answer *no*.

6 Mapping Reductions [20 points]

Recall that, if $w = a_1 \cdots a_n \in \Sigma^n$ is a string, $w^R = a_n \cdots a_1$ is the "reversal" of w. If $L \subseteq \Sigma^*$ is a language, $L^R = \{w^R : w \in L\}$. Let

$$\begin{aligned} A_{\mathrm{TM}} &= \{\langle M, w\rangle : M \text{ accepts } w\} \\ A_R &= \{\langle M\rangle : L(M) = (L(M))^R\} \end{aligned}$$

A. *Show that $A_{\mathrm{TM}} \leq_{\mathrm{m}} A_R$.*

We must map $\langle M, w\rangle$ into $\langle M'\rangle$ such that M accepts w iff $L(M') = (L(M'))^R$. The map must be Turing-computable. So let M' on input x behave as follows:

If $x = 01$ then accept.
Run M on w
If M accepts w, then accept
If M rejects w, then reject

Now if M accepts w then $L(M') = \Sigma^*$ so $L(M') = (L(M'))^R$; while if M does not accept w then $L(M') = \{01\}$ so $L(M') \neq (L(M'))^R$.

B. *Show that $\overline{A_{\mathrm{TM}}} \leq_{\mathrm{m}} A_R$.*

We must map $\langle M, w\rangle$ into $\langle M'\rangle$ such that (a) if M does not accept w then $L(M') = (L(M'))^R$, and (b) if M does accept w then $L(M') \neq (L(M'))^R$. The map must be Turing-computable. So let M' on input x behave as follows:

Run M on w
If M accepts w and $x = 01$ then accept
Reject

Now if M does not accepts w then $L(M') = \emptyset$ so $L(M') = (L(M'))^R$; while if M does accept w then $L(M') = \{01\}$ so $L(M') \neq (L(M'))^R$.

7 NP-Completeness [20 points]

Let $G = (V, E)$ be a graph. We say that $G' = (V', E')$ is a vertex-induced subgraph of G if $V' \subseteq V$ and E' is all the edges of G both endpoints of which are in V'. Now suppose each edge $e \in E$ as an integer weight, $w(e)$. Then the weight of the subgraph G' is just $\sum_{e' \in E'} w(e')$.

Show that the following language of graphs with heavy vertex-induced subgraphs is NP-Complete.

$$\mathit{HVIS} = \{\langle G, w, B\rangle : G = (V,E) \text{ is a graph, } w : E \to \mathsf{Z} \text{ specifies an integer weight, } w(e) \text{, for each } e \in E \text{, and } B \in \mathsf{Z} \text{ is an integer; and } G \text{ has some vertex-induced subgraph of weight at least } B.\}$$

First we show that HVIS $\in$ NP. Given $\langle G, w, B\rangle$, just "guess" a subset of vertices $V' \subseteq V$ and verify that the total weight of the subgraph induced by V' is at least B. That is, the certificate identifies the induced subgraph.

Now we show that CLIQUE $\leq_{\mathrm{P}}$ HVIS. Given a graph $G = (V, E)$ and a number k we must map it into a graph $G' = (V', E')$ and a weight function w and a bound B such that G has a clique of size k if and only iff G' has an induced subgraph of weight at least B, relative to w. So given $G = (V, E)$ and k, where $n = |V|$, let $G' = (V, E')$ be the complete graph on n-node. Let $w(e)$ to be 1 if $e \in E$ and $w(e) = -n^2$ otherwise. Let $B = \binom{k}{2} = k(k-1)/2$. Then if G has a k-clique W then W names a vertex-induced subgraph of weight $\binom{k}{2}$ and so the instance of HVIS we have generated is a yes-instance. Conversely, if G' has a vertex-induced subgraph of weight $B = \binom{k}{2}$ then the vertices of this subgraph must comprise a clique of size at least k because it can contain no non-edges of E, as even one such edge would result in the weight of the induced subgraph being negative. The transformation is clearly polynomial time, so we are done.