Notes on Computation Theory

Konrad Slind
`slind@cs.utah.edu`

September 21, 2010

To summarize, we have seen methods for translating between DFAs, NFAs, and regular expressions:

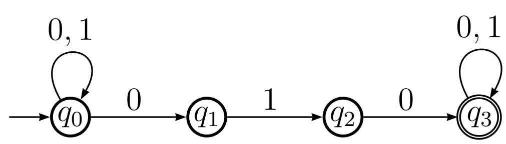
- Every DFA is an NFA.
- Every NFA can be converted to an equivalent DFA, by the subset construction.
- Every regular expression can be translated to an equivalent NFA, by the method in Section 5.4.2.
- Every DFA can be translated to a regular expression by the method in Section 7.1.3.

Notice that, in order to say that these translations work, *i.e.*, are correct, ’ we need to use the concept of formal language.

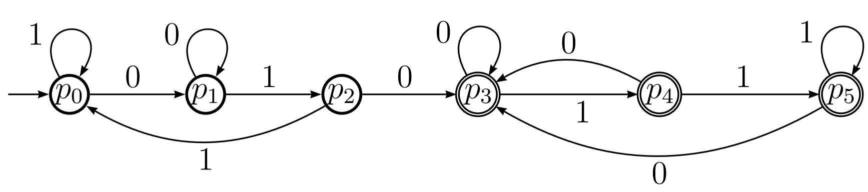
5.5 Minimization

Now we turn to examining how to reduce the size of a DFA such that it still recognizes the same language. This is useful because some transformations and tools will generate DFAs with a large amount of redundancy.

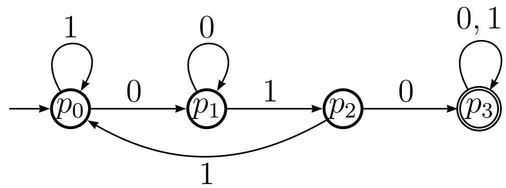
Example 122. Suppose we are given the following NFA:



The subset construction yields the following (equivalent) DFA:



which has 6 reachable states, out of a possible $2^4 = 16$. But notice that p_3, p_4, and p_5 are all accept states, and it's impossible to 'escape' from them. So you could collapse them to one big success state. Thus the DFA is equivalent to the following DFA with 4 states:



There are methods for systematically reducing DFAs to equivalent ones which are minimal in the number of states. Here's a rough outline of a minimization procedure:

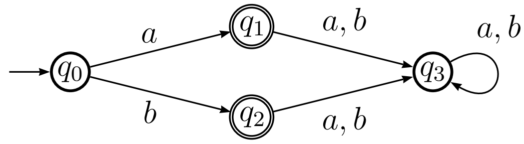
1. Eliminate inaccessible, or unreachable, states. These are states for which there is no string in Σ^* that will take the machine to that state.

   How is this done? We have already been doing it, somewhat informally, when performing subset constructions. The idea is to start in q_0 and mark all states accessible in one step from it. Now repeat this from all the newly marked states until no new marked state is produced. Any unmarked states at the end of this are inaccessible and can be deleted.

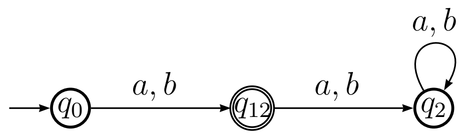
2. Collapse *equivalent* states. We will gradually see what this means in the following examples.

Remark. We will only be discussing minimization of DFAs. If asked to minimize an NFA, first convert it to a DFA.

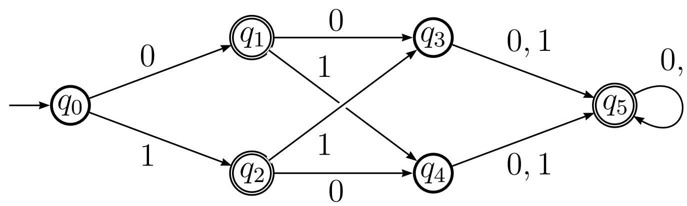
Example 123. The 4 state automaton



is clearly equivalent to the following 3 state machine:



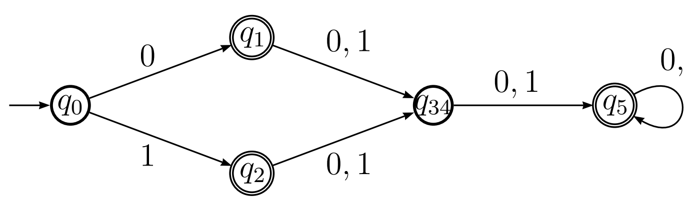
Example 124. The DFA



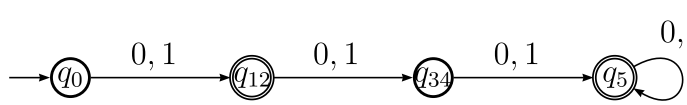
recognizes the language

$$\{0, 1\} \cup \{x \in \{0, 1\}^* \mid \mathsf{len}(x) \geq 3\}$$

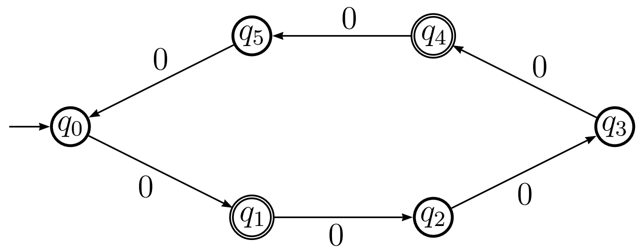
Now we observe that q_3 and q_4 are equivalent, since both go to q_5 on anything. Thus they can be collapsed to give the following equivalent DFA:



By the same reasoning, q_1 and q_2 both go to q_{34} on anything, so we can collapse them to state q_{12} to get the equivalent DFA



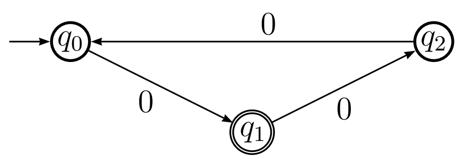
Example 125. The DFA



recognizes the language

$$\{0^n \mid \exists k.\, n = 3k + 1\}$$

This DFA minimizes to



How is this done, you may ask.

The main idea is a process that takes a DFA and combines states of it in a step-by-step fashion, where each steps yields an equivalent automaton. There are a couple of criteria that must be observed:

- We never combine a final state and a non-final state. Otherwise the language recognized by the automaton would change.
- If we merge states p and q, then we have to combine $\delta(p, a)$ and $\delta(q, a)$, for each $a \in \Sigma$. Contrarily, if $\delta(p, a)$ and $\delta(q, a)$ are not equivalent states, then p and q can not be equivalent.

Thus if there is a string $x = x_1 \cdot \ldots \cdot x_n$ such that running the automaton M from state p on x leaves M in an accept state and running M from state q on x leaves M in a non-accept state, then p and q cannot be equivalent. However, if, for all strings x in Σ^*, running M on x from p yields the same acceptance verdict (accept/reject) as M on x from q, then p and q are equivalent. Formally we define equivalence $\approx$ as

Definition 38 (DFA state equivalence)**.**

$$p \approx q \text{ iff } \forall x \in \Sigma^*.\ \Delta(p, x) \in F \text{ iff } \Delta(q, x) \in F$$

where F is the set of final states of the automaton.

Question: What is Δ?

Answer Δ is the extension of δ from symbols (single step) to strings (multiple steps). Its formal definition is as follows:

$$\begin{array}{rcl} \Delta(q, \varepsilon) & = & q \\ \Delta(q, a \cdot x) & = & \Delta(\delta(q, a), x) \end{array}$$

Thus $\Delta(q, x)$ gives the state after the machine has made a sequence of transitions while processing x. In other words, it's the state at the end of the computation path for x, where we treat q as the start state.

Remark. $\approx$ is an equivalence relation, *i.e.*, it is reflexive, symmetric, and transitive:

- $p \approx p$
- $p \approx q \Rightarrow q \approx p$
- $p \approx q \wedge q \approx r \Rightarrow p \approx r$

An equivalence relation partitions the underlying set (for us, the set of states Q of an automaton) into disjoint *equivalence classes*. This is denoted by $Q/\approx$. Each element of Q is in one and only one partition of $Q/\approx$.

Example 126. Suppose we have a set of states $Q = \{q_0, q_1, q_2, q_3, q_4, q_5\}$ and we define $q_i \approx q_j$ iff $i \bmod 2 = j \bmod 2$, *i.e.*, q_i and q_j are equivalent if i and j are both even or both odd. Then $Q/\approx = \{\{q_0, q_2, q_4\}, \{q_1, q_3, q_5\}\}$.

The equivalence class of $q \in Q$ is written $[q]$, and defined

$$[q] = \{p \mid p \approx q\} \ .$$

We have the equality

$$\underbrace{p \approx q}_{\text{equivalence of states}} \quad \text{iff} \quad \underbrace{([p] = [q])}_{\text{equality of sets of states}}$$

The *quotient* construction builds equivalence classes of states and then treats each equivalence class as a single state in the new automaton.

Definition 39 (Quotient automaton)**.** Let $M = (Q, \Sigma, \delta, q_0, F)$ be a DFA. The quotient automaton is $M/\!\approx\; = (Q', \Sigma, \delta', q_0', F')$ where

- $Q' = \{[p] \mid p \in Q\}$, *i.e.*, $Q/\!\approx$
- Σ is unchanged
- $\delta'([p], a) = [\delta(p, a)]$, *i.e.*, transitioning from an equivalence class (where p is an element) on a symbol a is implemented by making a transition $\delta(p, a)$ in the original automaton and then returning the equivalence class of the state reached.
- $q_0' = [q_0]$, *i.e.*, the start state in the new machine is the equivalence class of the start state in the original.
- $F' = \{[p] \mid p \in F\}$, *i.e.*, the set of equivalence classes of the final states of the original machine.

Theorem 18. *If M is a DFA that recognizes L, then $M/\!\approx$ is a DFA that recognizes L. There is no DFA that both recognizes L and has fewer states than $M/\!\approx$.*

OK, OK, enough formalism! we still haven't addressed the crucial question, namely how do we calculate the equivalence classes?

There are several ways; we will use a *table-filling* approach. The general idea is to assume initially that all states are equivalent. But then we use our criteria to determine when states are not equivalent. Once all the non-equivalent states are marked as such, the remaining states must be equivalent.

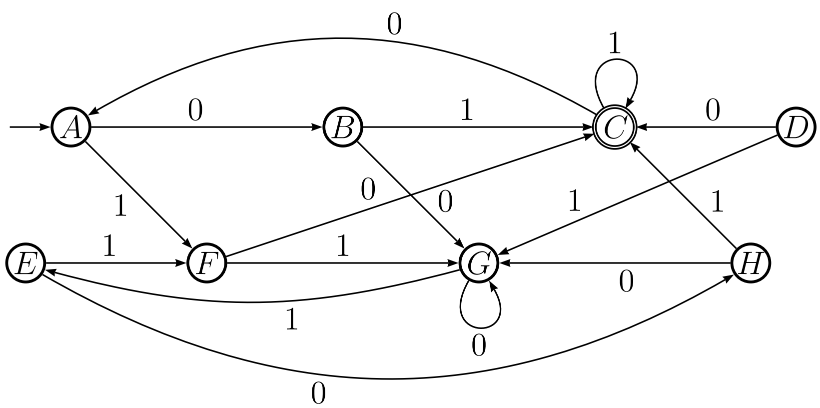
Consider all pairs of states p, q in Q. A pair p, q is *marked* once we know p and q are *not equivalent*. This leads to the following algorithm:

1. Write down a table for the pairs of states
2. Mark (p, q) in the table if $p \in F$ and $q \notin F$, or if $p \notin F$ and $q \in F$.
3. Repeat until no change can be made to the table:
   - if there exists an unmarked pair (p, q) in the table such that one of the states in the pair $(\delta(p, a), \delta(q, a))$ is marked, for some $a \in \Sigma$, then mark (p, q).

4. Done. Read off the equivalence classes: if (p, q) is not marked, then $p \approx q$.

Remark. We may have to revisit the same (p, q) pair several times, since combining two states can suddenly allow hitherto equivalent states to be markable.

Example 127. Minimize the following DFA



We start by setting up our table. We will be able to restrict our attention to the lower left triangle, since equivalence is symmetric. Also, each box on the diagonal will be marked with $\approx$, since every state is equivalent to itself. We also notice that state D is not reachable, so we will ignore it.

	A	B	C	D	E	F	G	H
A	$\approx$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
B		$\approx$	$-$	$-$	$-$	$-$	$-$	$-$
C			$\approx$	$-$	$-$	$-$	$-$	$-$
D	$-$	$-$	$-$	$-$	$-$	$-$	$-$	$-$
E				$-$	$\approx$	$-$	$-$	$-$
F				$-$		$\approx$	$-$	$-$
G				$-$			$\approx$	$-$
H				$-$				$\approx$

Now we split the states into final and non-final. Thus, a box indexed by p, q will be labelled with an X if p is a final state and q is not, or *vice versa*.

Thus we obtain

	A	B	C	D	E	F	G	H
A	$\approx$	–	–	–	–	–	–	–
B		$\approx$	–	–	–	–	–	–
C	X_0	X_0	$\approx$	–	–	–	–	–
D	–	–	–	–	–	–	–	–
E			X_0	–	$\approx$	–	–	–
F			X_0	–		$\approx$	–	–
G			X_0	–			$\approx$	–
H			X_0	–				$\approx$

State C is inequivalent to all other states. Thus the row and column labelled by C get filled in with X_0. (We will subscript each X with the step at which it is inserted into the table.) However, note that C, C is not filled in, since $C \approx C$. Now we have the following pairs of states to consider:

$$\{AB, AE, AF, AG, AH, BE, BF, BG, BH, EF, EG, EH, FG, FH, GH\}$$

Now we introduce some notation which compactly captures how the machine transitions from a pair of states to another pair of states. The notation

$$p_1p_2 \xleftarrow{0} q_1q_2 \xrightarrow{1} r_1r_2$$

means $q_1 \xrightarrow{0} p_1$ and $q_2 \xrightarrow{0} p_2$ and $q_1 \xrightarrow{1} r_1$ and $q_2 \xrightarrow{1} r_2$. If one of p_1, p_2, r_1, or r_2 are already marked in the table, then there is a way to distinguish q_1 and q_2: they transition to inequivalent states. Therefore $q_1 \not\approx q_2$ and the box labelled by q_1q_2 will become marked. For example, if we take the state pair AB, we have

$$BG \xleftarrow{0} AB \xrightarrow{1} FC$$

and since FC is marked, AB becomes marked as well.

	A	B	C	D	E	F	G	H
A	$\approx$	–	–	–	–	–	–	–
B	X_1	$\approx$	–	–	–	–	–	–
C	X_0	X_0	$\approx$	–	–	–	–	–
D	–	–	–	–	–	–	–	–
E			X_0	–	$\approx$	–	–	–
F			X_0	–		$\approx$	–	–
G			X_0	–			$\approx$	–
H			X_0	–				$\approx$

In a similar fashion, we examine the remaining unassigned pairs:

- $BH \xleftarrow{0} AE \xrightarrow{1} FF$. Unable to mark.
- $BC \xleftarrow{0} AF \xrightarrow{1} FG$. Mark, since BC is marked.
- $BG \xleftarrow{0} AG \xrightarrow{1} FE$. Unable to mark.
- $BG \xleftarrow{0} AH \xrightarrow{1} FC$. Mark, since FC is marked.
- $GH \xleftarrow{0} BE \xrightarrow{1} CF$. Mark, since CF is marked.
- $GC \xleftarrow{0} BF \xrightarrow{1} CG$. Mark, since CG is marked.
- $GG \xleftarrow{0} BG \xrightarrow{1} CE$. Mark, since CE is marked.
- $GG \xleftarrow{0} BH \xrightarrow{1} CC$. Unable to mark.
- $HC \xleftarrow{0} EF \xrightarrow{1} FG$. Mark, since CH is marked.
- $HG \xleftarrow{0} EG \xrightarrow{1} FE$. Unable to mark.
- $HG \xleftarrow{0} EH \xrightarrow{1} FC$. Mark, since CF is marked.
- $CG \xleftarrow{0} FG \xrightarrow{1} GE$. Mark, since CG is marked.
- $CG \xleftarrow{0} FH \xrightarrow{1} GC$. Mark, since CG is marked.
- $GG \xleftarrow{0} GH \xrightarrow{1} EC$. Mark, since EC is marked.

The resulting table is

	A	B	C	D	E	F	G	H
A	$\approx$	–	–	–	–	–	–	–
B	X_1	$\approx$	–	–	–	–	–	–
C	X_0	X_0	$\approx$	–	–	–	–	–
D	–	–	–	–	–	–	–	–
E		X_1	X_0	–	$\approx$	–	–	–
F	X_1	X_1	X_0	–	X_1	$\approx$	–	–
G		X_1	X_0	–		X_1	$\approx$	–
H	X_1		X_0	–	X_1	X_1	X_1	$\approx$

Next round. The following pairs need to be considered:

$$\{AE, AG, BH, EG\}$$

The previously calculated transitions can be re-used; all that will have changed is whether the 'transitioned-to' states have been subsequently marked with an X_1:

AE: unable to mark

AG: mark because BG is now marked.

BH: unable to mark

EG: mark because HG is now marked

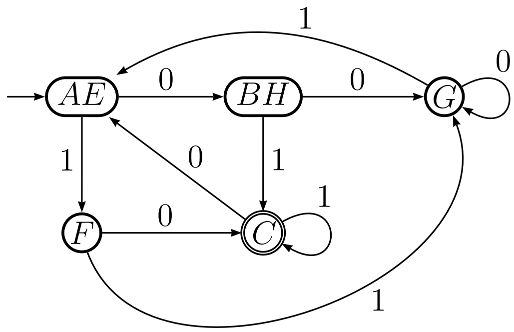
The resulting table is

	A	B	C	D	E	F	G	H
A	$\approx$	–	–	–	–	–	–	–
B	X_1	$\approx$	–	–	–	–	–	–
C	X_0	X_0	$\approx$	–	–	–	–	–
D	–	–	–	–	–	–	–	–
E		X_1	X_0	–	$\approx$	–	–	–
F	X_1	X_1	X_0	–	X_1	$\approx$	–	–
G	X_2	X_1	X_0	–	X_2	X_1	$\approx$	–
H	X_1		X_0	–	X_1	X_1	X_1	$\approx$

Next round. The following pairs remain: $\{AE, BH\}$. However, neither makes a transition to a marked pair, so the round adds no new markings to the table. We are therefore done. The quotiented state set is

$$\{\{A, E\}, \{B, H\}, \{F\}, \{C\}, \{G\}\}$$

In other words, we have been able to merge states A and E, and B and H. The final automaton is given by the following diagram.



5.6 Decision Problems for Regular Languages

Now we will discuss some questions that can be asked about automata and regular expressions. These will tend to be from a general point of view, *i.e.*., involve arbitrary automata. A question that takes any automaton (or collection of automata) as input and asks for a terminating algorithm yielding a boolean (true or false) answer is called a *decision problem*, and a program that correctly solves such a problem is called a *decision algorithm*. Note well that a decision problem is typically a question about the (often infinite) set of strings that a machine must deal with; answers that involve running the machine on every string in the set are not useful, since they will take forever. That is not allowed: in every case, a decision algorithm must return a correct answer in finite time.

Here is a list of decision problems for automata and regular expressions:

1. Given a string x and a DFA M, $x \in \mathcal{L}(M)$?
2. Given a string x and an NFA N, $x \in \mathcal{L}(N)$?
3. Given a string x and a regular expression r, $x \in \mathcal{L}(r)$?
4. Given DFA M, $\mathcal{L}(M) = \emptyset$?
5. Given DFA M, $\mathcal{L}(M) = \Sigma^*$?
6. Given DFAs M_1 and M_2, $\mathcal{L}(M_1) \cap \mathcal{L}(M_2) = \emptyset$?