Graph Theory Definitions

1. A *graph* is an ordered tuple $G = (V, E)$, where V is a nonempty set which we call vertices, and E is a set (possibly empty) of 2-element subsets of V, which we call edges.

   Example 1: Let $G = (V, E)$, and let $V = \{1, 2, 3, 4\}, E = \{\{1, 2\}, \{2, 3\}\}$. Then by our definition, G is a graph.

   Example 2: Let $G = (V, E)$, and let $V = \{\emptyset\}$. G is not a graph, because we constrain V to be nonempty.

   Example 3: Let $G = (V, E)$, and let $V = \{1, 2, 3\}$ and $E = \{\{1, 2\}, \{3, 4\}\}$. Then G is not a graph, because we expect each element of E to be a subset of V, and $4 \notin V$. Similarly, $E = \{\{1, 2, 3\}\}$ is not allowed because each element of E is not a 2-element subset of V.

   Because we are using sets and not multisets, $V_1 = \{1, 2, 3\}$ and $V_2 = \{1, 2, 2, 3\}$ are indistinguishable. Duplicate elements are considered to be the same elements (perhaps written twice). For clarity, we only write them once. If we were using multisets, then V_1 and V_2 would be different. The sets we use are not ordered sets; therefore $e_1 = \{1, 2\}$ is the same edge as $e_2 = \{2, 1\}$; they are just different ways of denoting the same edge. Sometimes E may be defined to be multiset (to allow for parallel edges), and may even contain 2-element multisets (self-loops), but in this class, we do not allow parallel edges or self-loops by our definition of a graph.[1]

2. A *subgraph*[2] of $G = (V, E)$ is a graph $G_1 = (V_1, E_1)$ such that $V_1 \subseteq V$ and $E_1 \subseteq E$.

3. A *path* is a sequence of vertices $v_1 v_2 \cdots v_t$ such that $\forall i, 1 \leq i < t$, $\{v_i, v_{i+1}\} \in E$.

4. A graph $G = (V, E)$ is *connected* if for every pair of vertices $v, w \in V$, there is a path a path $v_1 v_2 \cdots v_t$ such that $v = v_1$ and $w = v_t$.

5. The sets of nodes that are reachable from one another impose an equivalence relation. We call the subgraphs consisting of the nodes that are reachable from one another and the edges incident to those nodes a *component*. In other words, a component $C = (V_C, E_C)$ is a subgraph of G such that $\forall u, w \in V_C$, there is a path from u to w. The edges that are incident to those vertices comprise E_C. Note that in different components, the intersection of their vertex sets and their edge sets are empty. Therefore, the components partition V and E.

   An alternate definition of a connected graph is that it has only one component.

6. A *cycle* of graph $G(V, E)$ is a path of 3 or more vertices such that $\{v_1, v_t\} \in E$.

7. A graph is *acyclic* if it has no cycle.

8. A *tree* is a connected acyclic graph.

9. A *weighted graph* is a graph G together with a weighting function $w : E \to \mathbb{R}$.

[1] Definitions are like standards this way; the problem is, there are so many of them.
[2] This one wasn't presented in class, but I thought I'd add it in.

10. Let G be a connected graph, $G = (V, E)$. A *spanning tree* of G is a set $E_T \subseteq E$ such that every vertex $v \in V$ is incident to some edge $e \in E_T$, and $T = (V, E_T)$ is a tree.
11. **Facts about trees:**
    (a) A tree of n vertices always has $n - 1$ edges.
    (b) Add any edge to a tree, it makes a unique cycle.
    (c) Remove any edge from that cycle and it's a tree again!
12. A connected graph of $n - 1$ edges is a tree.
13. A *forest* is a collection of trees. In other words, a forest is an acyclic graph.