Midterm Solutions

Problems 1–10 were together worth 75 point. For grading, 0–4 correct was 0 points; 5 correct was 5 points; 6 correct was 15 points; 7 correct was 30 points; 8 correct was 45 points; 9 correct was 60 points; 10 correct was 75 points.

I have added parenthetical explanations for you. They were not asked for in the problem.

1. *The logical connectives* $\{\wedge, \vee\}$ *are logically complete.* **false**
(This was a homework problem. The idea was that, with "and" ($\wedge$) and "'or" ($\vee$) alone, you can never make the negation ($\neg$) functionality.
2. *If a Boolean formula is tautological then it is satisfiable.* **true**
(A formula is tautological if it is true under *every* truth assignment (t.a.). To be satisfiable is a weaker requirement—that it be true under *some* t.a.)
3. *If* ϕ *is Boolean formula then either* ϕ *or* $\neg\phi$ *is tautological.* **false**
(If $\phi = A$, say, neither ϕ or $\neg\phi$ is tautological. Read the English carefully; it doesn't ask if $(\phi \vee \neg\phi)$ is tautological, which it is; it asks if ϕ is tautological or $\neg\phi$ is.)
4. *Any Boolean formula can be realized (that is, equivalently written) using just NAND gates.* **true**
(We explained in class that NAND, like NOR, is logically complete)
5. *If* A *and* B *are finite sets then* $|A \cup B| = |A| + |B|$. **false**
(This is inclusions/exclusion gone awry: the correct formula is $|A \cup B| = |A| + |B| - |A \cap B|$. So the initial claim is true iff A and B are disjoint.)
6. *If* A, B, C *are sets,* $x \in A \oplus B \oplus C$ *iff* x *is in exactly one of* A, B, *and* C. **false**
(A point x is in the xor of of a bunch of sets iff it is in an odd number of them. So, for a counterexample, put a point x in all three sets, eg, $A = B = C = \{1\}$, whence $A \oplus B \oplus C = \{1\}$.)
7. *An infinite language is one that contains at least one string of infinite length.* **false**
(Strings are never infinite in this class; we have never considered any such thing. An infinite language is one that contains an infinite number of strings. Each of those infinitely many strings will have finite length.)
8. $\bigcup_{i\in\mathbb{N}} \{1^i\} = 1^*$ *(where the right-hand side is interpreted as a language). (Recall that* $\mathbb{N} = \{0, 1, 2, \ldots\}$ *is the set of natural numbers and* $1^0 = \varepsilon$*).* **true**
(On the left, we throw into one pot every string of 1's, including the empty string. On the right, we do the same. I attempted to resolve any unclarity about whether or not I was including the emptystring on the left by way of the second parenthetical comment. Note that both conventions are common: $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathbb{N} = \{1, 2, 3, \ldots\}$, so I was *attempting* to be clear. Maybe I failed, as a student told me after lecture that I had earlier employed the opposite convention. I'm afraid that I have the memory of a fruit fly, and assume everyone else does, too; that is why I try to define what I mean nearby.)
9. *Let* T *be the relation on pairs of people defined by saying that* $x\,T\,y$ *iff* x *is at least as tall as* y*. Then* T *is an equivalence relation.* **false**
(The relation "at least as tall as" is reflexive and transitive, but it's definitely not symmetric. You need all three to be an equivalence relation.)
10. *In PS #1 you were asked to show that there exist irrational* a *and* b *such that* a^b *is rational. The distributed solution I gave out showed this to be the case for* $a = b = \sqrt{2}$. **false**
(Look back at the the solution to this homework problem. I explicitly commented that the proof was *non*-constructive—we showed that *either* $\sqrt{2}^{\sqrt{2}}$ was irrational or something else was. (In fact, $\sqrt{2}^{\sqrt{2}}$ *is* irrational, but I doubt that anyone managed to show this in their homework.)

The remaining problems were graded 10 points per problem.

11. *Convert the following formula into an equivalent one without negations. The logical connectives your formula may use are* $\{\wedge, \vee\}$. *The relation symbols your formula may use are* $\{<, \leq, ,=, \neq, \geq, >\}$. *As always, be careful.*

$$\neg\,(\forall c)(\exists N)(\forall n)((c > 0 \wedge N > 0 \wedge n \geq N) \rightarrow f(n) \leq n^{-c})$$

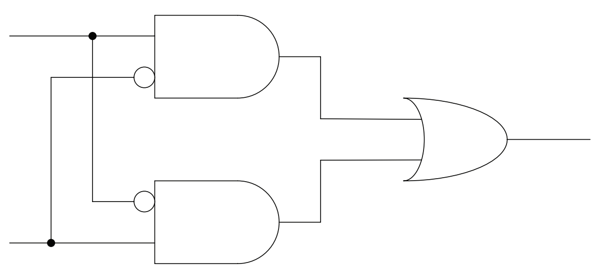
The negation is: $(\exists c)(\forall N)(\exists n)(c > 0 \wedge N > 0 \wedge n \geq N \wedge f(n) > n^{-c})$.

12. *Write a* ***disjunctive normal form (DNF)*** *formula whose truth table is given below. Let your formula be the* ***or*** *of four terms where each term is the* ***and*** *of three variables or their complements:*

A	B	C	$\phi(A,B,C)$
0	0	0	0
0	0	1	1
0	1	0	1
0	1	1	0
1	0	0	1
1	0	1	0
1	1	0	1
1	1	1	0

$\overline{A}\,\overline{B}\,C \vee \overline{A}\,B\,\overline{C} \vee A\,\overline{B}\,\overline{C} \vee A\,B\,\overline{C}$

13. *Using only* ***and***, ***or***, *and* ***not*** *gates, draw a circuit realizing the functionality of an* ***xor*** *(exclusive or) gate.*



Of course various other circuits are possible, too.

14. *Capture the logical content of the following sentence in a Boolean formula: Nobody likes Mark except his roommates, who actually do like him. You may use any logical connectives you know. Make your formula as succinct as possible. Use predicate symbols* $L(x, y)$ *(person* x *likes person* y*),* $R(x, y)$ *(persons* x *and* y *are roommates), and the constant symbol* Mark. *The universe of discourse is people.*

$(\forall x)(L(x, \mathsf{Mark}) \Leftrightarrow R(x, \mathsf{Mark}))$

(This is of course a repeat of a problem from one of your quizzes.)

15. *Suppose that* A, B, A', *and* B' *are sets. Give a simple counter-example to the claim:*
$A \times B = A' \times B' \implies A = A' \wedge B = B'$.

A simple counterexample is $A = B = A' = \emptyset$ and $B' = \{\emptyset\}$

16. Let $L = \{x : x \in \{a, b, c\}^*$ and x contains at least one a and at least one $b\}$. *Write, in lexicographic order, the first five strings of* L.

ab, *ba*, *aab*, *aba*, *abb*

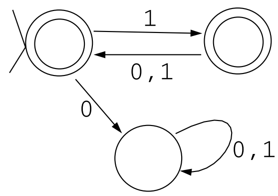
17. *Finish this statement of DeMorgan's law for sets:* $(A \cap B)^c =$ $A^c \cup B^c$.

    (Many students wrote $\neg A$ for A^c. In fact, when A is a set, we have never defined a "$\neg A$". People do use $\overline{A}$ for the set-theoretic complement of A, but I've never seen $\neg A$.)

18. *Write a shortest regular expression for the language* $L = \{a^n : n \text{ is odd}\}$.

    $a(aa)^*$

19. *Draw a DFA for* $L = \{a_1 \dots a_n : \text{ each } a_i \text{ is a bit and } a_i = 1 \text{ for all odd } i\}$*. Assume that the empty string* ε *is in* L*. Make your DFA have the minimum number of states you can.*



    (This was of course a homework problem.)

20. *Let* $P \subseteq A \times B$ *and* $Q \subseteq B \times C$ *be a relation. Formally define* $P \circ Q$*, the composition of* P *and* Q*.*

    $P \circ Q$ is the relation $P \circ Q \subseteq A \times C$

    defined by: $a(P \circ Q)c$ iff $(\exists b \in B)(a\,P\,b \wedge b\,Q\,c)$

21. *For* $a, b \in \mathbb{R}$ *define* $a\ E\ b$ *if* $a - b \in \mathbb{Z}$*. What's the smallest positive number in* $[\pi]$*? (Here* $\pi = 3.14159\cdots$ *is the usual constant that goes by this name.)*

    $\pi - 3$

    (Strangely few students got this. You should have been familiar with this equivalence relation, which was used in a homework problem.)

22. *Let* T_n *be the minimum number of moves to solve the* n*-ring tower of Hanoi problem. In class we showed that, for* $n \geq 1$,

$$T_n \quad \leq \quad 2T_{n-1} + 1 \qquad (1)$$

(while $T_0 = 0$*). Repeating the proof given in class, establish this equation. Please do not "solve" the equation (that is, write it in a "closed-form" way); your job is to justify it. Write your proof in clear, grammatical English.*

The problem is asking you to show that equation (1) gives a *sufficient* number of moves to solve the problem, so it is enough to *give an algorithm* that solves the problem using $T_n = 2T_{n-1} + 1$ moves. An algorithm is:

> (i) move the top $n-1$ rings off of the peg they're on and onto a temporary peg; (ii) then move the largest ring to the destination peg; (iii) then move the $n-1$ rings from the temporary peg to the destination peg.

The first and third steps are accomplished recursively, doing nothing at all when $n = 0$. The total number of moves in this solution is $S_n = S_{n-1} + 1 + S_{n-1} = 2S_{n-1} + 1$ (for $n \geq 1$) and $S_0 = 0$, so $T_n \leq S_n$ and we are done.

Note: in many students' solutions it was impossible to figure out if you even understood that you were being asked to prove that the formula is giving a *necessary* number of moves or a *sufficient* number of moves. When you use language like *you must now do X*, it certainly sounds like you are arguing necessity (because of *must*). Also, many students seemed to completely ignore the instructions about not solving the recurrence.

23. *We showed in class that five shuffles is inadequate to mix a deck a 52 cards. In a clearly written paragraph, recount the line of reasoning for our proof.*

    We defined a *rising sequence* as a maximal subsequence of consecutive integers then defined $\mathrm{nr}(S)$ as the number of rising sequences needed to "cover" the sequence S. Letting $S_0 = (1, 2, \ldots, 52)$ and $S_* = (52, 51, \ldots, 1)$, we commented that $\mathrm{nr}(S_0) = 1$ while $\mathrm{nr}(S_*) = 52$. We showed that if S' is obtained from S by a single riffle shuffle then $\mathrm{nr}(S') \leq 2 \cdot \mathrm{nr}(S)$—that is, the number of rising sequences at most doubles with each shuffle. So, if you start with the deck in the order given by S_0 then, after 5 shuffles, you can't possibly be in any configuration having more than $2^5 = 32$ rising sequences so, in particular, you can't be in configuration S_*. If you can't even reach every ordering of cards then clearly then the cards are not well mixed.