Problem Set 7 — Due Friday, May 26, 2000

1. Prove that, for every $n \geq 1$, $n^3 + 2n$ is divisible by 3.
2. Prove that, for every $n \geq 1$, $(1 + 1/2)^n \geq 1 + n/2$.
3. Recall that $A \oplus B = (A - B) \cup (B - A)$. For $n \geq 3$, let $\oplus_{i=1}^{n} A_i = (\oplus_{i=1}^{n-1} A_i) \oplus A_n$. Prove that for any $n \geq 2$, $A_1 \oplus \cdots \oplus A_n$ consists of those elements in an odd number of the sets $A_1, \ldots, A_n$.
4. Prove that for every integer $n \geq 2$ the number of lines obtained by joining n distinct points in the plane, no three of which are collinear, is $n(n-1)/2$.
5. Prove that for any integer $n \geq 1$, if $x_1, \ldots, x_n$ are distinct real numbers, then no matter how the parentheses are inserted into their product, the number of multiplications used to compute the product is $n - 1$.

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6. Prove that $\sum_{i=1}^{n} i^2 = n(n+1)(2n+1)/6$
7. Recall the fair cake-cutting protocol described in lecture. For $n = 2$ people to share the cake, person 1 cuts the cake evenly in half, and person 2 then chooses the larger piece. For $n \geq 3$ people to share the cake, person n steps aside; persons $1, \ldots, n-1$ use the cake-cutting algorithm (recursively) to cut the cake into $n-1$ pieces; each person $1, \ldots, n-1$ cuts his piece into n equal pieces; person n chooses the largest piece from each of the pieces now in possession of persons $1, \ldots, n-1$; persons $1, \ldots n-1$ each get to keep his remaining $n-1$ pieces.

   (a) Write a recurrence relation for $T(n)$, the number of cuts used by the algorithm to split the cake among n persons.

   (b) Compute $T(6)$.

   (c) Find a closed-form solution for $T(n)$.

8. For $n \geq 1$, let $B(n)$ be the number of ways to express n as the sum of 1s and 2s, taking order into account. Thus $B(4) = 5$ because $5 = 1+1+1+1 = 1+1+2 = 1+2+1 = 2+1+1 = 2+2$.

   (a) Compute $B(i)$ for $1 \leq i \leq 5$ by showing all the different ways to write these numbers as above.

   (b) Find a recursive definition for $B(n)$ and identify this sequence.

   (c) Compute $B(10)$.

9. Solve the following recurrence relation: $T(n) = 3T(n-2) + 1$ if $n \geq 2$; and $T(0) = T(1) = 0$.
10. Solve the following recurrence relation: $T(n) = nT(n-1)$ when $n \geq 2$, and $T(1) = 1$.
11. Solve the following recurrence relation to within a $\Theta(\cdot)$ result. Assume that $T(n) \in \Theta(1)$ for sufficiently small n. The recurrence is: $T(n) = 2T(n/5) + n$.
12. Same instructions as the last problem; the recurrence is $T(n) = 5T(n/2) + n$.
13. Same instructions as the last problem; the recurrence is $T(n) = 5T(n/5) + n$.
14. Form a group of 13 men, 6 women, 2 boys, and 4 girls. (a) How many ways can a man, a woman, a boy, and a girl be selected? (b) How many ways can a male and a female be selected? (c) How many ways can a person be selected?
15. How many numbers in the range 100–999 have no repeated digits? Now how many of them are even? (Be careful.)
16. How many different functions are there from 4-byte words to 4-byte words?