EAD210A- Numerical Methods _ Rao Vemuri

Homework Assignment 2. (Due 14 October 2004)

Programming assignment (35 points)

1. (20 pts) Consider the functions e^x and |x|.
   a. Plot on [-1. +1] the Lagrange interpolation polynomials of degree N for N = 2, 4, 6, 8, 10 of these functions with $x_i = -1 + \frac{2i}{N}, i = 0,1,..., N$ (overlay your plots)
   b. Plot on [-1. +1] the Hermite interpolation polynomials of degree N for N = 3, 5 of these functions with $x_i = -1 + \frac{2i}{N}, i = 0,1,..., N$
2. (15 pts) Show that the interpolation polynomials of $f(x) = \frac{1}{1+x}$ on [0,1] at the points $x_i = i/N$, i = 0,1,…,N converges to f by plotting the interpolation polynomials for N = 10, 20.

Paper and Pencil assignment (65 points)

3. (15 pts) *Interpolation on product spaces*: Suppose every linear interpolation problem stated in terms of functions $\varphi_1, \varphi_2, ..., \varphi_N$ has a unique solution

$$\Phi(x) = \sum_{i=0}^{N} \alpha_i \varphi_i(x)$$

with $\Phi(x_k) = f_k, k = 0,1,2,..., N$ for prescribed support arguments $x_0, x_1, ..., x_N$ with $x_i \neq x_j, i \neq j$. Show the following. If $\psi_0, \psi_1, ..., \psi_M$ is also a set of functions for which every linear interpolation problem has a unique solution, then for every choice of abscissas

$$x_0, x_1, ..., x_N; x_i \neq x_j, i \neq j$$
$$y_0, y_1, ..., y_M; y_i \neq y_j, i \neq j$$

and support ordinates

$$f_{ik}, i = 0,1,..., N; k = 0,1,..., M$$

there exists a unique function of the form

$$\Phi(x, y) = \sum_{\nu=0}^{N} \sum_{\mu=0}^{M} \alpha_{\nu\mu} \varphi_\nu(x) \psi_\mu(y)$$

with $\Phi(x_i, y_k) = f_{ik}, i = 0,1,2,..., N; k = 0,1,..., M$.

4. (15 pts) Continuation of Hw#1, Problem #10 (Bessel function problem). Compare the result obtained in (10b) of HW#1, with the behavior of the error

$$\max_{0\le x\le 1} | S_{\Delta_N}(x) - J_0(x) |$$

as $N \to \infty$, where S_{Δ_N} is the interpolating spline function with the knot set $\Delta_N = \{x_i^N\}$ and $S'_{\Delta_N}(x) = J'_0(x)$, for x = 0, 1.

5. (15 pts) On page 68 of the text, carry out the missing steps leading to Eq (1.6-2) and Eq. (1.6-3).

6. (20 points) Define the spline function S_j for equidistant knots $x_i = a + ih, h > 0, i = 0,1,..., N$ by

$$S_j(x_k) = \delta_{jk};\ j,k = 0,1,...,N \quad \text{and} \quad S''_j(x_0) = S''_j(x_N) = 0$$

Verify that the moments $m_1, m_2, ..., m_{N-1}$ of S_j are:

$$m_i = -\frac{1}{\rho_i} m_{i+1}, i = 0,1,..., j-2$$

$$m_i = -\frac{1}{\rho_{N-i}} m_{i-1}, i = j+2,..., N-1$$

$$m_j = -\frac{6}{h^2} \frac{2 + (1/\rho_{j-1}) + (1/\rho_{N-j-1})}{4 - (1/\rho_{j-1}) - (1/\rho_{N-j-1})}, \quad j \neq 0,1, N-1, N$$

$$m_{j-1} = -\frac{1}{\rho_{j-1}} (6h^{-2} - m_j), \qquad j \neq 0,1, N-1, N$$

$$m_{j+1} = -\frac{1}{\rho_{N-j-1}} (6h^{-2} - m_j), \qquad j \neq 0,1, N-1, N$$

where the numbers ρ_j are recursively defined by

$$\rho_1 := 4$$

$$\rho_i := 4 - \frac{1}{\rho_{i-1}}, i = 2,3,...$$

Hint: To get you started, you may wish to consult the text book. At the beginning of the derivation, the book noted that the second derivative of the spline function coincides with a linear function in each sub-interval $[x_j, x_{j+1}]$, $j = 0,\ldots N-1$, in terms of the moments. Indeed, the second derivative evaluated at the grid point x_j is the moment m_j. You follow the derivation in the book until you get the tridiagonal matrix. The solution of that should match the above equations.