EGR2013 Tutorial 8

Linear Algebra

Outline

- **Powers of a Matrix and Matrix Polynomial**
- **Vector Algebra**
- **Vector Spaces**

Powers of a Matrix and Matrix Polynomial

If A is a square matrix, then we define the nonnegative integer power of A to be

$$A^0 = I$$

$$A^n = AA\cdots A$$

If A is invertible, then we define the negative integer powers to be

$$A^{-n} = (A^{-1})^n = A^{-1}A^{-1}\cdots A^{-1}$$

If A is a square matrix, we have a matrix polynomial in A as

$$p(A) = a_0 I + a_1 A + a_2 A^2 + \cdots + a_n A^n$$

<u>Example 1</u>

If A is a n-th order square matrix and $A^k = 0$, evaluate $(I - A)^{-1}$.

Solution:

$$\because A^k = 0$$

$$\therefore I - A^k = I$$

$$\because I - A^k = (I - A)(I + A + A^2 + \cdots + A^{k-1}) = I$$

$$\therefore (I - A)^{-1} = I + A + A^2 + \cdots + A^{k-1}$$

Definition of Vectors

We often use two kinds of quantities, namely scalars and vectors.
A scalar is a quantity that is determined by its magnitude;
A vector is a quantity that is determined by both its magnitude and its direction.

Equality of Vectors: two vectors **a** and **b** are equal, if they have the same length and the same direction.

Representations: in Cartesian coordinate system, the vector can be described using real numbers. If the given vector **a** has an initial point $P_1(x_1, y_1, z_1)$ and a terminal point $P_2(x_2, y_2, z_2)$, then the vector **a** can be described as

$$a = \overrightarrow{P_1P_2} = [x_2 - x_1, y_2 - y_1, z_2 - z_1]$$

And the norm $|a| = \sqrt{a_1^2 + a_2^2 + a_3^2}$.

Another representation of vectors is $a = [a_1 \quad a_2 \quad a_3] = a_1\mathrm{i} + a_2\mathrm{j} + a_3\mathrm{k}$,

where **i, j, k** are the standard unit vectors in the positive directions of the axes in a Cartesian coordinate system.

Basic Properties of Vectors:

- **Zero vector 0** has length 0 and no direction.
- **Negative vector –a** has the length $|a|$ and the direction is opposite to that of **a**.
- **Unit vector** is a vector of norm 1

Vector Addition and Scalar Multiplication

If $a = [a_1 \quad a_2 \quad a_3]$ and $b = [b_1 \quad b_2 \quad b_3]$ are two vectors, then

$$a \pm b = [a_1 \pm b_1 \quad a_2 \pm b_2 \quad a_3 \pm b_3]$$
$$ka = [ka_1 \quad ka_2 \quad ka_3]$$

Basic Properties:

- **a+b=b+a**   **(a+b)+c=a+(b+c)**
- **a+0=0+a=a**   **a+(-a)=0**
- **k(a+b)=ka+kb**   **(k+l)a=ka+la**
- **k(la)=(kl)a**   **0a=0**
- **1a=a**   **(-1)a=-a**

Inner Product or Dot Product

<u>Definition:</u> if **a** and **b** are two vectors and θ is the angle between **a** and **b**, then the dot product or Euclidean inner product $a \bullet b$ is defined by

$$a \bullet b = |a||b|\cos\theta = a_1 b_1 + a_2 b_2 + a_3 b_3$$

<u>Angle between two vectors:</u>

The angle θ between the vectors is $\cos\theta = \dfrac{a \bullet b}{|a||b|}$.

<u>Orthogonality:</u>
The inner product of two vectors is zero if and only if these vectors are perpendicular.

<u>Properties of the Dot Product:</u>

- Schwarz inequality: $|a \bullet b| \le |a||b|$
- Triangle inequality: $|a \bullet b| \le |a| + |b|$

<u>Orthogonal Projection:</u>
Orhtogonal projection of a vector **a** in the direction of a vector **b** is defined by

$$p = proj_b \mathrm{a} = \frac{\mathrm{a} \bullet \mathrm{b}}{|\mathrm{b}|^2}\mathrm{b}$$

<u>Example 2</u>

If vector **a** is perpendicular to any vector, show **a** is a zero vector.
<u>Proof:</u>

$\because$ **a** is perpendicular to any vector

$\therefore$ for any vector β, $a \bullet \beta = 0$

especially, if $\beta = a$, thus $a \bullet a = 0$

$\therefore |a| = 0$

therefore, **a** is a zero vector.

Zero vector is perpendicular to any vector.

Example 3

Show vector **c** and vector $(a \bullet c)b-(b \bullet c)a$ are orthogonal vectors.

Proof:

$$
\begin{aligned}
&[(a \bullet c)b-(b \bullet c)a] \bullet c \\
&=[(a \bullet c)b] \bullet c-[(b \bullet c)a] \bullet c \\
&=(a \bullet c)(b \bullet c)-(b \bullet c)(a \bullet c) \\
&=0
\end{aligned}
$$

Thus **c** and $(a \bullet c)b-(b \bullet c)a$ are orthogonal vectors.

Vector Product or Cross Product

Definitions: if **a** and **b** are two vectors, then the cross product or vector product is a vector $v = a \times b = [v_1 \quad v_2 \quad v_3]$

The vector **v** can be obtained from the expansion by the first row of the symbolical third-order determinant

$$
v=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}=\begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} i-\begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} j+\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} k
$$

The length of the vector **v** is given as $|a \times b|=|a||b|\sin\theta$

The direction of the vector **v** is perpendicular to both **a** and **b**.

Example 4

If two vectors **a** and **b**, $|a| = 10$, $|b| = 2$ and $a \bullet b = 12$, Evaluate $|a \times b|$.

Solution:

$$
\begin{aligned}
&\because a \bullet b=|a||b|\cos\theta \\
&\therefore \cos\theta = \frac{a \bullet b}{|a||b|} = \frac{3}{5} \\
&\therefore \sin\theta = \frac{4}{5} \\
&\therefore |a \times b| = |a||b|\sin\theta = 16
\end{aligned}
$$

Example 5

Given $a \times b = c \times d, a \times c = b \times d$, show $a-d$ is parallel to $b-c$.

Proof:

If $a-d$ is parallel to $b-c$, then $(a-d) \times (b-c)$ should be **0.**

$$\therefore (a-d) \times (b-c) = a \times b + d \times c - d \times b - a \times c$$
$$= a \times b - c \times d + b \times d - a \times c = 0$$

therefore, vector $a-d$ is parallel to vector $b-c$.

Scalar Triple Product

If **a, b** and **c** are three vectors, then the scalar triple product is defined by

$$a \bullet (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

The absolute of the scalar triple product is the volume of the parallelepiped with **a, b** and **c** as edge vectors.

Example 6

Given three vectors **a**=**i+4j-4k, b=-5i+5j+k** and **c=-6i+j+5k**, show these three vector are in the plane.

Proof:

Because the absolute of the scalar triple product is the volume of the parallelepiped, if the volume is zero, we can say the edge vectors are in the same plane.

Thus, $a \bullet (b \times c) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} 1 & 4 & -4 \\ -5 & 5 & 1 \\ -6 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 4 & -4 \\ 0 & 25 & -19 \\ 0 & 25 & -19 \end{vmatrix} = 0$

Therefore, vectors **a, b** and **c** are in the same plane.

- ❖ If $a \bullet b = 0$, then vector **a** is perpendicular to vector **b;**
- ❖ If $a \times b = 0$, then vector **a** is parallel to vector **b;**
- ❖ If $a \bullet (b \times c) = 0$, then vectors **a, b** and **c** are in the same plane.

Vector Spaces

In a nonempty set V, we define two algebraic operations:

(1) vector addition: $\forall \alpha, \beta \in V \Rightarrow \alpha + \beta \in V$

(2) scalar multiplication: $\forall \alpha \in V, \forall k \in R \Rightarrow k\alpha \in V$

For vector addition:
- a+b=b+a
- (a+b)+c=a+(b+c)
- there is an object 0 in V, such that 0+a=a for all a in V
- for each a in V, there is an object –a in V, such that a+(-a)=0

For scalar multiplication:
- k(a+b)=ka+kb
- (k+l)a=ka+la
- k(la)=(kl)a
- for every a in V, 1a=a

If V follows all the axioms above, then V is called a vector space.

Example 6

Let $V = R^2$ and we define the addition and scalar multiplication in V for two vectors $a = [a_1 \quad a_2]$ and $b = [b_1 \quad b_2]$ and a scalar k as follows:

$$a + b = [a_1 \quad a_2] + [b_1 \quad b_2] = [a_1 + b_1 \quad 0]$$

$$ka = k[a_1 \quad a_2] = [ka_1 \quad 0]$$

First, $\forall \alpha, \beta \in V \Rightarrow \alpha + \beta \in V$ and $\forall \alpha \in V, \forall k \in R \Rightarrow k\alpha \in V$ are satisfied.

But there is no zero vector in V, such that $[a_1 \quad a_2] + [? \quad ?] = [a_1 \quad a_2]$

So set V is not a vector space with stated operations.