EGR2013 Tutorial 9

Linear Algebra

Outline

- **Linear Independence/Dependence of Vectors and Subspace**
- **Fundamental Vector Spaces of a Matrix**
- **Matrix Eigenvalue Problem**

Linear Independence/Dependence of Vectors

A Linear combination of vectors $a_1, a_2, \cdots, a_m$ in a vector space V is an expression

$$k_1 a_1 + k_2 a_2 + \cdots + k_m a_m$$

These vectors are called linear independence if the vector equation

$$k_1 a_1 + k_2 a_2 + \cdots + k_m a_m = 0$$

implies that $k_1 = 0, k_2 = 0, \cdots, k_m = 0$. Otherwise, they are linear dependent.

Span: If $S = \{a_1, a_2, \cdots, a_r\}$ is a set of vectors in a vector space V, then the subspace W of V is called the space spanned by $a_1, a_2, \cdots, a_r$, denoted by

$$W = span(S) \quad \text{or} \quad W = span\{a_1, a_2, \cdots, a_r\}$$

Dimension: dim(V) = the maximum number of linear independent vectors.

Basis:

- The basis of V consists of a maximum possible number of linear independent vectors in V.
- Every vector in V can be expressed as a linear combination of the vectors in the basis.
- A basis is not unique.

Example 1

Given All vectors in R^4 such that $v_1 = 0, v_3 = 0, v_2 + v_4 \le 0$, is this set of vectors a vector space?

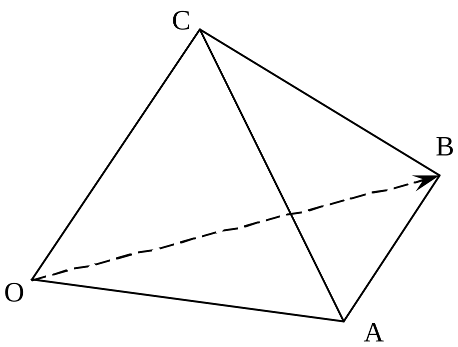
Example 2

If vectors a_1, a_2 and a_3 are linear independent, determine whether the vectors (1) $a_1 - a_2, a_2 - a_3, a_3 - a_1$; (2) $a_1 + a_2, a_2 + a_3, a_3 + a_1$ form a linear dependent set or a linear independent set.

Solution:

Geometric interpretation:

Because a_1, a_2 and a_3 are linear independent, so they are not in the same plane. We can assume they are the edges of a triangular pyramid O-ABC,



If $OA = a_1, OB = a_2, OC = a_3$, then $a_1 - a_2, a_2 - a_3$ and $a_3 - a_1$ are the edges of the triangle ABC respectively, so they are in the same plane, in other words, they are linear dependent. But $a_1 + a_2, a_2 + a_3$ and $a_3 + a_1$ are in the plane OAB, OBC and OCA respectively, obviously, they are not in the a plane, so they are linear independent.

(1) $a_1 - a_2, a_2 - a_3$ and $a_3 - a_1$ are linear dependent.

Because we have $1(a_1 - a_2) + 1(a_2 - a_3) + 1(a_3 - a_1) = 0$.

(2) $a_1 + a_2, a_2 + a_3$ and $a_3 + a_1$ are linear independent.

The vector equation

$$k_1(a_1 + a_2) + k_2(a_2 + a_3) + k_3(a_3 + a_1) = 0$$

can be written as

$$(k_1 + k_3)a_1 + (k_1 + k_2)a_2 + (k_2 + k_3)a_3 = 0$$

because a_1, a_2 and a_3 are linear independent, then we have

$$\begin{cases} k_1 & & + & k_3 & = & 0 \\ k_1 & + & k_2 & & = & 0 \\ & & k_2 & + & k_3 & = & 0 \end{cases}$$

because the determinant $\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = 2 \neq 0$

thus the system has trivial solutions $k_1 = k_2 = k_3 = 0$, so $a_1 + a_2, a_2 + a_3$ and $a_3 + a_1$ are linear independent.

Fundamental Vector Spaces of a Matrix

A is a $m \times n$ matrix, then we define,

Row space: subspace of R^m spanned by row vectors;

Column space: subspace of R^n spanned by column vectors;

Nullspace: the solution space of $AX = 0$.

Basic Concepts of Fundamental Matrix Space:

- Row operation doesn't change the fundamental spaces of a matrix;
- Basis can be obtained by reducing the matrix A to its echelon form;
- dim (row space) = dim (column space) = rank (A);
- The dimension of the nullspace of A is called the nullity of A, and rank(A)+nullity(A)=n, n is the number of column of A.

Example 3

A is a n-th order matrix, if A is invertible $\Leftrightarrow |A| \neq 0 \Leftrightarrow$ rank (A)=n

Example 4

Given rank(A)=3, evaluate a and b. $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & b \\ 2 & 3 & a & 4 \\ 3 & 5 & 1 & 7 \end{bmatrix}$

Solution:
Row operations:

$$A \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & b \\ 0 & 1 & a-2 & 2 \\ 0 & 2 & -2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & b \\ 0 & 0 & a-1 & 2-b \\ 0 & 0 & 0 & 4-2b \end{bmatrix}$$

because rank (A)=3, therefore $a \neq 1, b = 2$ or $a = 1, b \neq 2$.

Example 5

Given A is a $m \times n$ matrix, B is a $n \times s$ matrix, if AB=0, show $rank(A) + rank(B) \leq n$

Proof:

Matrix B can be expressed as $B = (B_1, B_2, \cdots, B_s)$, then we have

$$AB = A(B_1, B_2, \cdots, B_s) = (AB_1, AB_2, \cdots, AB_s) = (0, 0, \cdots, 0)$$

$$\Rightarrow AB_j = 0 \qquad j = 1, 2, \cdots, s.$$

Therefore, each column of B is a solution of AX=0. The dimension of nullspace of A is equal to $n - rank(A)$, B_j is some vectors in this nullspace, so $rank(B) = rank(B_1, B_2, \cdots, B_s) \leq n - rank(A)$.

Then we have $rank(A) + rank(B) \leq n$

Matrix Eigenvalue Problem

Eigenvalue and Eigenvetor:

For an $n \times n$ matrix A, and a vector x, if a value of l exists such that $Ax = l\,x$, has nonzero solution vector $x \neq 0$, then l is defined as an eigenvalue of matrix A, the corresponding solution vector x is eigenvector.

Method for Finding Eigenvalues and Eigenvectors:

- Characteristic equation: $p(l) = \det(A - l\,I) = (-1)^n\, l^{\,n} + \cdots + c_n = 0$
- The eigenvector corresponding to l is the solutions of $(A - l\,I)x = 0$

Example 6

$A=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, find the eigenvalues and eigenvectors of A.

Solution:

(1) The Characteristic equation of A is

$$\det(A-lI)=0\Rightarrow \begin{vmatrix} -l & 1 \\ -1 & -l \end{vmatrix}=l^2+1=0$$

$$\Rightarrow (l+i)(l-i)=0\Rightarrow l_1=-i,\ l_2=i$$

(2) Eigenvectors are the solutions of $(A-lI)x=0$

For $l_1=-i$, the linear system becomes

$$\begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0\Rightarrow \begin{bmatrix} i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=s\begin{bmatrix} i \\ 1 \end{bmatrix} \text{ so that } \begin{bmatrix} i \\ 1 \end{bmatrix}$$

is a eigenvector corresponding $l_1=-i$.

For $l_1=i$, the linear system becomes

$$\begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0\Rightarrow \begin{bmatrix} -i & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=0\Rightarrow \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}=s\begin{bmatrix} -i \\ 1 \end{bmatrix} \text{ so that } \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

is a eigenvector corresponding $l_1=i$.

Example 7

If l is the eigenvalue of matrix A, evaluate the eigenvalue of A^2+2A-I.

Solution:

If l is the eigenvalue of matrix A, and X is the eigenvector corresponding l,

then we have $Ax=lx$.

Multiply both sides by matrix A, then we get

$$AAx=Alx\Rightarrow A^2x=lAx=l^2x$$

thus l^2 is eigenvalue of matrix A^2 with the corresponding eigenvector x.

In the same way, we can get the eigenvalue of A^2+2A-I is l^2+2l-1.