EGR2013 Tutorials 6

Linear Algebra

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Linear Algebra

Example 1

A system of linear equations is shown below

$$\begin{cases} 2x+5y+6z=7 \\ 5x-3y+4z=0 \\ -7x+y+z=4 \end{cases}$$

We can use a matrix to indicate the equations

$$\begin{bmatrix} 2 & 5 & 6 & 7 \\ 5 & -3 & 4 & 0 \\ -7 & 1 & 1 & 4 \end{bmatrix}$$

The matrix shows all the information required to solve the equations. The solutions can be obtained by performing appropriate operations on this matrix. This method is particularly used in computer programs to solve the system of linear equations.

Definitions of Matrices

A **matrix** a rectangular array of numbers enclosed in brackets. The **size** of matrix is described in terms of number of rows (horizontal lines) and columns (vertical lines) it contains.
Some special kinds of matrix: **column matrix** (or column vector); **row matrix** (or row vector); **square matrix**; **rectangular matrix**. There are enough examples in the lecture notes and textbook.

Equality of Matrix

Tow matrix A=B if and only if they have the same size and the corresponding entries are equal.

Example 2

$$A = \begin{bmatrix} 1 & y & z \\ 2 & 6 & x \\ 3 & 4 & 5 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 9 & 8 \\ 2 & 6 & 7 \\ 3 & 4 & 5 \end{bmatrix}$$

If A=B, then x=7, y=9 and z=8. A can never equal to C if the size of C is $m \times n$ ($m \neq 3$ or $n \neq 3$).

Matrix Addition/Subtraction and Scalar Multiplication

If A and B are matrices of the same size, then A+B is equal to adding the entries of B to the corresponding entries of A; A-B is equal to subtracting the entries of B from the corresponding entries of A.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix} \quad B = \begin{bmatrix} b_{11} & b_{12} & b_{13} & \cdots & b_{1n} \\ b_{21} & b_{22} & b_{23} & \cdots & b_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ b_{m1} & b_{m2} & b_{m3} & \cdots & b_{mn} \end{bmatrix}$$

$$C = A \pm B = \begin{bmatrix} a_{11} \pm b_{11} & a_{12} \pm b_{12} & a_{13} \pm b_{13} & \cdots & a_{1n} \pm b_{1n} \\ a_{21} \pm b_{21} & a_{22} \pm b_{22} & a_{23} \pm b_{23} & \cdots & a_{2n} \pm b_{2n} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ a_{m1} \pm b_{m1} & a_{m2} \pm b_{m2} & a_{m3} \pm b_{m3} & \cdots & a_{mn} \pm b_{mn} \end{bmatrix}$$

Matrices of different sizes can not be added or subtracted.
The matrix addition or subtraction has the following properties

- A+0=A
- A-A=0
- A+B=B+A
- (A+B)+C=A+(B+C)

If A is any matrix and c is any scalar, then the product cA is the matrix obtained by multiplying each entry of the matrix A by c. The matrix cA is called a scalar multiple of A.

The scalar multiplication has the following properties

- 1A=A
- c(kA)=(ck)A
- c(A+B)=cA+cB
- (c+k)A=cA+kA

Example 3

$$A = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 & -2 \\ 4 & 5 \end{bmatrix} \quad C = \begin{bmatrix} 3 & 0 & 2 \\ 4 & 0 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 6 & 1 & -5 \\ 5 & -2 & 13 \end{bmatrix}$$

Find the following expressions or give reasons why they are undefined.

- 4A, A-B, A+B+C, C+D

Matrix Multiplication

If A is a $m \times n$ matrix and B is a $r \times p$ matrix, then product of C=AB exist if and only if $n = r$. C is a $m \times p$ matrix with entries:

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj} = a_{i1} b_{1j} + a_{i2} b_{2j} + \cdots + a_{in} b_{nj}$$

The matrix multiplication has the following properties

- $AB \neq BA$
- AB=0 does not necessarily imply A=0 or B=0 or BA=0
- AC=AD does not necessarily imply C=D (even when $A \neq 0$)
- (kA)B=k(AB)=A(kB), k is any scalar
- A(BC)=(AB)C
- (A+B)C=AC+BC
- C(A+B)=CA+CB

Example 4

$$A=\begin{bmatrix}1 & -2 & 3\\ 2 & -2 & -1\end{bmatrix} \quad B=\begin{bmatrix}1 & 2\\ 1 & 1\\ 2 & 0\end{bmatrix}$$

$$AB=\begin{bmatrix}1 & -2 & 3\\ 2 & -2 & -1\end{bmatrix}\begin{bmatrix}1 & 2\\ 1 & 1\\ 2 & 0\end{bmatrix}=\begin{bmatrix}5 & 0\\ -2 & 2\end{bmatrix}$$

$$BA=\begin{bmatrix}1 & 2\\ 1 & 1\\ 2 & 0\end{bmatrix}\begin{bmatrix}1 & -2 & 3\\ 2 & -2 & -1\end{bmatrix}=\begin{bmatrix}5 & -6 & 1\\ 3 & -4 & 2\\ 2 & -4 & 6\end{bmatrix}$$

Example 5

$$\begin{cases} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1\\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ a_{m1}x_1+a_{m2}x_2+\cdots+a_{mn}x_n=b_m \end{cases}$$

Let

$$A=\begin{bmatrix}a_{11} & a_{12} & \cdots & a_{1n}\\ a_{21} & a_{22} & \cdots & a_{2n}\\ \vdots & \vdots & \vdots & \vdots\\ a_{m1} & a_{m2} & \cdots & a_{mn}\end{bmatrix}, \quad X=\begin{bmatrix}x_1\\ x_2\\ \vdots\\ x_n\end{bmatrix}, \quad b=\begin{bmatrix}b_1\\ b_2\\ \vdots\\ b_m\end{bmatrix}$$

The above system of equations can be represented by **AX=b**

Matrix Transposition

The **transpose** A^T of an $m\times n$ matrix is the $n\times m$ matrix result from interchanging the rows and columns of A.

The transposition has the following properties

- $(A^T)^T=A$

- $(cA)^T = cA^T$
- $(A+B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$

Proof:

if $A = (a_{ij})_{m\times r}$, $B = (b_{ij})_{r\times n}$, $AB = (c_{ij})_{m\times n}$, then $(AB)^T = (c_{ji})_{n\times m}$

The entry c_{ji} at the jth row and ith column of $(AB)^T$ is the same as the entry at the ith row and jth column of AB, equals $\sum_{k=1}^{r} a_{ik} b_{kj}$.

The entry c_{ji} of $B^T A^T$ is obtained by adding up the products of each entry in jth row of B^T and corresponding entry in ith column of A^T, the jth row of B^T is the jth column of B and the ith column of A^T is the ith row of A. Then c_{ji} equals $\sum_{k=1}^{r} b_{kj} a_{ik}$.

So $(AB)^T = B^T A^T$.

Special Matrices

- Symmetric Matrices: $A^T = A$. (i.e. AA^T or $A^T A$)
- Skew-symmetric Matrices: $A^T = -A$. (i.e. $A - A^T$)
- Upper Triangular matrices
- Lower triangular matrices
- Diagonal matrices

Vector

Definition: A vector is a matrix that has one row (row vector) or one column (column vector).

Inner Product of Vector

a: a row vector; b: a column vector, the same components. $a \bullet b$ is the inner product or dot product.

$$a \bullet b = \begin{bmatrix} a_1 & a_2 & \cdots & a_n \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \sum_{k=1}^{n} a_k b_k = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$$

System of Linear Equations

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{cases}$$

The system of linear equations can be written as AX=b. A is the **coefficient matrix**.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

The augmented matrix is $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_n \end{bmatrix}$

if all bi are zero, it's a **homogeneous** system, otherwise it's called a **nonhomogeneous** system.
Every system of linear equations has either
(a) no solutions
(b) exactly one solution
(c) infinite solutions

Gaussian Elimination

The aim of Gaussian Elimination is to reduce the augmented matrix to an echelon form by performing the row operations.
Row operations include
- interchange one row with another
- add multiples of one row to another

Example 6

Solve the following linear equations by Gauss Elimination.

$$\begin{cases} 2x_1 & + & x_2 & + & 3x_3 & = & 1 \\ 4x_1 & + & 2x_2 & + & 5x_3 & = & 4 \\ 2x_1 & & & + & 2x_3 & = & 6 \end{cases}$$

Solution:
The augmented matrix for this system is

$$\begin{bmatrix} 2 & -1 & 3 & 1 \\ 4 & 2 & 5 & 4 \\ 2 & 0 & 2 & 6 \end{bmatrix}$$

Add -2 times the top row to the second row
Add -1 times the top row to the last row

$$\begin{bmatrix} 2 & -1 & 3 & 1 \\ 0 & 4 & -1 & 2 \\ 0 & 1 & -1 & 5 \end{bmatrix}$$

Interchange the second row and the last row

$$\begin{bmatrix} 2 & -1 & 3 & 1 \\ 0 & 1 & -1 & 5 \\ 0 & 4 & -1 & 2 \end{bmatrix}$$

Add -4 times the second row to the last row

$$\begin{bmatrix} 2 & -1 & 3 & 1 \\ 0 & 1 & -1 & 5 \\ 0 & 0 & 3 & -18 \end{bmatrix}$$

The system is changed into

$$\begin{cases} 2x_1 & - & x_2 & + & 3x_3 & = & 1 \\ & & x_2 & - & x_3 & = & 5 \\ & & & & 3x_3 & = & -18 \end{cases}$$

Using the back-substitution method, we can readily get the solution

$$x_3 = -6, x_2 = -1, x_1 = 9$$