Theorem:

Let $P(n)$ be a proposition that depends on a natural number n.

If we can show:

- $P(n_0)$ is true, n_0 natural number
- $P(n) \to P(n+1)$ is true, $n \geq n_0$

then the method of proof by induction allows us to conclude that $P(n)$ is true for all $n \geq n_0$.

Example: Let k be a natural number.

We define the sequence a_k by:

$$\begin{cases} a_k = a_{k-1} + k + 4 & \text{for } k \geq 2 \\ a_1 = 5 \end{cases}$$

Show that $a_k = \dfrac{k(k+9)}{2}$ for all $k \geq 1$.

Let us define

$$b(k) = \frac{k(k+9)}{2}$$

$P(k)$: $\quad a_k = b(k), \quad k \geq 1$.

<u>basis step:</u> $k = 1$

$a_1 = 5$

$b(1) = \frac{1(1+9)}{2} = \frac{10}{2} = 5$

$> P(1)$ is true.

<u>Inductive step:</u> $P(k) \rightarrow P(k+1) \quad k \geq 1$

I assume $P(k)$ is true: $a_k = b(k)$

$a_{k+1} = \quad ?$

$$b(k+1) = \frac{(k+1)(k+10)}{2}$$

I know $a_k = a_{k-1} + k + 4$

$a_{k+1} = a_{k+1-1} + k+1 + 4$

$$a_{k+1} = a_k + k+5$$

$$= b(k) + k+5$$

$$= \frac{k(k+9)}{2} + k+5$$

$$= \frac{k(k+9) + 2k+10}{2}$$

$$= \frac{k^2 + 11k + 10}{2}$$

$$b(k+1) = \frac{(k+1)(k+10)}{2} = \frac{k^2+10k+k+10}{2}$$

$$= \frac{k^2+11k+10}{2}$$

$a_{k+1} = b(k+1)$: $P(k+1)$ is true.

The method of proof by induction allows me to conclude that $P(k)$ is true, for all $k \geq 1$.

Exercise:

Find a closed form expression for the sum $S(n) = \frac{2}{3} + \frac{2}{3^2} + \cdots \frac{2}{3^n}$

$$S(n) = \sum_{i=1}^{n} \frac{2}{3^i}$$

n	$S(n)$
1	$\frac{2}{3} = \frac{3-1}{3}$
2	$\frac{2}{3} + \frac{2}{9} = \frac{2 \times 3 + 2}{9} = \frac{8}{9} = \frac{3^2 - 1}{3^2}$
3	$\frac{8}{9} + \frac{2}{27} = \frac{8 \times 3 + 2}{27} = \frac{26}{27} = \frac{3^3 - 1}{3^3}$
4	$\frac{26}{27} + \frac{2}{81} = \frac{3 \times 26 + 2}{81} = \frac{80}{81} = \frac{3^4 - 1}{3^4}$

$$S(n) = \frac{3^n - 1}{3^n} = 1 - \frac{1}{3^n}$$

Let us define the Proposition:

$$P(n): \quad S(n) = 1 - \frac{1}{3^n}$$

We want to show $P(n)$ is true for all $n \geq 1$.

We define $A(n) = 1 - \frac{1}{3^n}$

Basis step: $n = 1$

$$S(1) = \frac{2}{3}$$

$$A(1) = 1 - \frac{1}{3} = \frac{2}{3}$$

$\Rightarrow P(1)$ is true.

Inductive step: $P(n) \longrightarrow P(n+1) \quad n \geq 1$.

I assume $P(n)$ is true: $S(n) = A(n)$.

$$S(n+1) = \frac{2}{3} + \frac{2}{3^2} + \cdots \frac{2}{3^n} + \frac{2}{3^{n+1}}$$

$$= S(n) + \frac{2}{3^{n+1}}$$

$$= A(n) + \frac{2}{3^{n+1}}$$

$$= 1 - \frac{1}{3^n} + \frac{2}{3^{n+1}} = 1 - \frac{3}{3^{n+1}} + \frac{2}{3^{n+1}} =$$

$$= 1 + \frac{2-3}{3^{n+1}} = 1 - \frac{1}{3^{n+1}}$$

$A(n+1) = 1 - \frac{1}{3^{n+1}}$

$P(n+1)$ is true.

The method of proof by induction allows me to conclude that $P(n)$ is true for all $n \geq 1$.

Exercise 3: Let F_n be the Fibonacci numbers,
Show that F_{3n} is even, for all $n \geq 1$.

$P(n)$: F_{3n} is even.

Proof by induction:

Basis step: $n = 1$

$$F_3 = F_2 + F_1 = 1 + 1 = 2, \text{ even.}$$

$P(1)$ is true.

Inductive step: $P(n) \rightarrow P(n+1)$ $n \geq 1$.

We assume $P(n)$ is true, F_{3n} is even

There exists an integer k such that $F_{3n} = 2k$.

$$F_{3(n+1)} = F_{3n+3} = F_{3n+2} + F_{3n+1}$$

$$= F_{3n+1} + F_{3n} + F_{3n+1}$$

$$= 2F_{3n+1} + 2k = 2[F_{3n+1} + k]$$

$F_{3(n+1)}$ is even, $P(n+1)$ is true.