<u>Proof by induction:</u>

Let $P(n)$ be a proposition that depends on a natural number n.

If we can show:

- $P(n_0)$ is true, n_0 natural number
- $P(n) \to P(n+1)$ is true, $n \geq n_0$

Then the method of proof by induction allows me to conclude that $P(n)$ is true for all $n \geq n_0$.

<u>Example:</u> Let n be a natural number. We define the sequence a_n by:

$$\begin{cases} a_n = a_{n-1} + n + 4 & \text{when } n \geq 2 \\ a_1 = 5 \end{cases}$$

Show that $a_n = \dfrac{n(n+9)}{2}$ for all $n \geq 1$.

Proof by induction:

1. Basis step:

$n = 1$

$$a_1 = 5$$

let us define $\quad b_n = \dfrac{n(n+9)}{2}$

$$b_1 = \frac{1(1+9)}{2} = 5$$

$a_1 = b_1$: $\quad P(1)$ is true.

2. $P(n) \rightarrow P(n+1) \qquad n \geq 1$

I assume $P(n)$ is true, i.e.,

$$a_n = b_n$$

~~$a_{n+1} =$~~ I know $\quad a_n = a_{n-1} + n + 4$

$$a_{n+1} = a_{n+1-1} + n + 1 + 4$$

$$a_{n+1} = a_n + n + 5$$

$$a_{n+1} = a_n + n + 5$$

but $\quad a_n = b_n = \dfrac{n(n+9)}{2}$

$$a_{n+1} = \frac{n(n+9)}{2} + n + 5$$

$$= \frac{n(n+9) + 2(n+5)}{2}$$

$$= \frac{n^2 + 9n + 2n + 10}{2}$$

$$a_{n+1} = \frac{n^2 + 11n + 10}{2}$$

$$b_{n+1} = \frac{(n+1)(n+1+9)}{2}$$

$$= \frac{(n+1)(n+10)}{2} = \frac{(n^2 + 10n + n + 10)}{2}$$

$$= \frac{n^2 + 11n + 10}{2}$$

$a_{n+1} = b_{n+1}$: $P(n+1)$ is true.

The method of proof by induction allows me to conclude that $P(n)$ is true for all $n \geq 1$.

Exercice 2

Find a closed form for the sum

$$S_n = \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3} + \cdots \frac{2}{3^n}$$

n	S_n	Guess
1	$\frac{2}{3}$	$\frac{3^1-1}{3^1}$
2	$\frac{2}{3}+\frac{2}{3^2}=\frac{2}{3}+\frac{2}{9}=\frac{8}{9}$	$\frac{3^2-1}{3^2}$
3	$\frac{8}{9}+\frac{2}{3^3}=\frac{8}{9}+\frac{2}{27}=\frac{26}{27}$	$\frac{3^3-1}{3^3}$

$$P(n): \quad S_n = \frac{3^n-1}{3^n}$$

We prove that $P(n)$ is true when $n \geq 1$ using a proof by induction.

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$$S_n = \frac{2}{3} + \frac{2}{3^2} + \cdots \frac{2}{3^n}$$

$$A_n = \frac{3^n - 1}{3^n}$$

$P(n)$: $S_n = A_n$

Basis step: $n = 1$

$$S_1 = \frac{2}{3}$$

$$A_1 = \frac{3-1}{3} = \frac{2}{3}$$

$>$ $P(1)$ is true

Inductive step: $P(n) \to P(n+1)$ $n \geq 1$

We assume $P(n)$ is true, $S_n = A_n$

$$S_{n+1} = \frac{2}{3} + \frac{2}{3^2} + \cdots \frac{2}{3^n} + \frac{2}{3^{n+1}}$$

$$= S_n + \frac{2}{3^{n+1}}$$

$$= A_n + \frac{2}{3^{n+1}}$$

$$= \frac{3^n - 1}{3^n} + \frac{2}{3^{n+1}} = \frac{3(3^n - 1) + 2}{3^{n+1}}$$

$$= \frac{3^{n+1} - 1}{3^{n+1}}$$

$$A_{n+1} = \frac{3^{n+1} - 1}{3^{n+1}}$$

$P(n+1)$ is true.

The method of proof by induction allows me to conclude that $P(n)$ is true for $n \geq 1$

Exercise 3

Let F_n be the Fibonacci numbers.

Show that F_{3n} is even for all $n \geq 1$.

Proof by induction: $P(n)$: F_{3n} is even

Basis step: $n = 1$.

$$F_3 = F_2 + F_1 = 1 + 1 = 2 \text{ is even.}$$

$P(1)$ is true.

Inductive step: $P(n) \to P(n+1) \quad n \geq 1$

I assume $P(n)$ is true: F_{3n} is even, there exists an integer k such that $F_{3n} = 2k$.

$$\begin{aligned} F_{3(n+1)} &= F_{3n+3} \\ &= F_{3n+2} + F_{3n+1} \\ &= F_{3n+1} + F_{3n} + F_{3n+1} \\ &= 2F_{3n+1} + 2k = 2(\underbrace{F_{3n+1} + k}_{\text{integer}}) \end{aligned}$$

F_{3n+3} is even: $P(n+1)$ is true.