Chapter 9
Finding a Line Transversal of Axial Objects in Three Dimensions

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Abstract

An axial object in E^3 is a box or rectangle, all of whose edges are parallel to the coordinate axes. A line transversal of a set of axial objects is a line that intersects every object. We present an algorithm which finds a line transversal, if one exists, in expected linear time. In the process, we generalize a randomized linear programming algorithm, and prove that the set of line transversals of axial objects has a constant number of connected components.

1 Introduction

A line which intersects every member of a given set of objects is a *line transversal* (or *stabbing line*) for the set. Let us call a line in E^3 that is parallel to either the x,y or z axis an *axial* line. An *axial box* or an *axial rectangle* is one whose edges are all segments of axial lines. In this paper, we give an algorithm to find a line transversal of a set of n axial boxes and axial rectangles in E^3, in expected $O(n)$ time.

The general dimensional version of this problem arises in statistics, when one wants to find a linear approximation for data given by ranges in each dimension [Pon 91]. The three dimensional case is particularly important in computer graphics, when one wants to find a line of sight through a sequence of rectangular windows or holes [Tel 91].

Our algorithm improves an $O(n \lg n)$ expected time algorithm of Hohmeyer and Teller [HT91]. For the more general problem of finding a line transversal for a set of polyhedra with a constant number of edge directions, Pellegrini gives an $O(n^2 \lg n)$ algorithm [Pel 90]. For general polyhedra in E^3, the problem of finding a line transversal was studied in [AW87], [MO88], and [PS90], where they give an $O(n^3 + 2^{\sqrt{\lg n}})$ algorithm to find all extremal line transversals.

Here, we reduce the problem to a more general one involving directed lines in E^3, which we solve using a generalization of Seidel's randomized linear programming algorithm [RS 90]. We give a set of geometric conditions under which Seidel's algorithm works, and show that this problem satisfies the conditions.

We also prove a geometric theorem of independent interest. For general polyhedra, the set of line transversals can have $\Omega(n^2)$ connected components [Pel 90]. We show that the set of line transversals of a set of axial objects has a constant number of connected components.

2 The Algorithm

Here we present a generalized version of Seidel's linear programming algorithm, and give some geometric conditions under which it runs correctly in expected $O(n)$ time. The input to the generalized algorithm is an ordered set O of points and a finite set L of constraints. In linear programming, O is R^d and L is a set of linear halfspaces. Every $m \in O$ and $l \in L$ are related by a subroutine $side(l,m)$ which takes values $\{+1,0,-1\}$. We say that m is *feasible* with respect to l when $side(l,m) \geq 0$, and m is *on* l when $side(l,m) = 0$. The algorithm searches for the minimal point m which is feasible with respect to every l, by trying to partition L into two sets B and L', such that the minimum feasible m is on all $l \in B$. A second subroutine $min(B)$ returns the minimum m which is on all $l \in B$, if such a point exists. Observe that for $m = min(B)$, every $l \in B$ is on m.

```
initialize B = {}
algorithm GLP(B, L)
    if L = B then
        m = min(B) /* base case */
    else
        l = RandomElement(L)
        m = GLP(B, L - {l})
        if side(l, m) < 0
            m = GLP(B ∪ {l}, L - {l})
    return m
```

Now we describe a set of geometric conditions on the sets O and L, under which this algorithm can be applied. A *path* is a continuous mapping of the closed interval $[0,1]$ to O. Let $m_0, m_1 \in O$ be two points. From now on, witout loss of generality let $m_1 > m_0$.

CONDITION 1: For every $m_0, m_1 \in 0$, there is a *canon-*

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ical path $path(m_0, m_1)$, which satisfies:
CROSSING CONDITION: If, for any $l \in L$, $side(l, m_1) \geq 0$ and $side(l, m_0) \leq 0$, then for some $m \in path(m_0, m_1)$, $side(l, m) = 0$.
ENDPOINT CONDITION: For any $m \in path(m_0, m_1)$, $m_1 \geq m \geq m_0$.
CONVEXITY CONDITION For any $l \in L$, either $side(l, m) = 0$ for every $m \in path(m_0, m_1)$, or $side(l, m) = 0$ for at most one $m \in path(m_0, m_1)$.

CONDITION 2: There is some constant k, such that, for any $A \subseteq L$, if $min(A)$ exists, then there are at most k constraints $l \in A$ such that $min(A - \{l\}) < min(A)$.

In linear programming, the canonical paths are line segments, and the constant $k = d$.

THEOREM 2.1. *For any pair of sets O and L which satisfy conditions 1 and 2, algorithm GLP finds a minimum $m \in O$ feasible with respect to every $l \in L$, if one exists, in expected time $O(k!n)$.*

Proof. We use the notation

$$C(B) = \{m : side(l, m) = 0, \forall l \in B\}$$

First we show that $GLP(B, L)$ correctly finds the minimum m, if one exists, such that $m \in C(B)$ and m is feasible with respect to L. This is true when $L = \{\}$. Now assume, for the induction, that $m_0 = GLP(B, L - \{l\})$ is correct. If $side(l, m_0) \geq 0$, then $m_0 = GLP(B, L)$. If $side(l, m_0) < 0$, we show by contradiction that the correct m_1 is on l. If not, then some $m \in path(m_0, m_1)$ is on l because of the Crossing Condition, so m is feasible with respect to l. Since both $m_0, m_1 \in C(B)$, m must also be in $C(B)$ by the Convexity Condition. This m is also feasible with respect to every $l' \in L - \{l\}$, by the Convexity Condition. And $m < m_1$, by the Endpoint Condition. This contradicts the assumption that m_1 is the correct $GLP(B, L)$, so m_1 must be on l. Therefore $m_1 = GLP(B \cup \{l\}, L - \{l\}) = GLP(B, L)$. Since the algorithm is correct for any B, it is correct for $B = \{\}$, and returns the minimum feasible $m \in O$.

For the time bound, note that we are guaranteed that at every level of recursion $GLP(B, L)$ will be on every $l \in B$. Because of condition 2, the probability that the removal of l changes the minimum is $\leq (k - |B|)/n$. Condition 2 also implies that it is always true that $|B| \leq k$, since an object only gets added to B if its removal changes the minimum. Assuming that $min(B)$ can be computed in time $O(|B|)$, and that $side(l, m)$ can be computed in constant time, we get the recurrence

$$T(n, k) \leq T(n-1, k) + O(1) + \frac{k}{n} T(n-1, k-1)$$

whose solution is $O(k!n)$ [RS 90].

We will now show that finding a line transversal of axial objects can be reduced to a geometric problem which satisfies conditions 1 and 2.

3 Linespace

We will reduce the problem of finding a line transversal of axial objects to a problem involving directed lines in E^3. Before we can state the new problem, we need to give some definitions and describe some geometric properties of directed lines.

Let (p_x, p_y, p_z) be a point in E^3. An ordered pair of distinct points determines a directed line. *Linespace* is the space of directed lines in E^3.

Let l and m be two directed lines, defined by points (l^1, l^2) and (m^1, m^2) respectively. Define $side(l, m)$ as the sign of the determinant

$$\begin{vmatrix} l_x^1 & l_y^1 & l_z^1 & 1 \\ l_x^2 & l_y^2 & l_z^2 & 1 \\ m_x^1 & m_y^1 & m_z^1 & 1 \\ m_x^2 & m_y^2 & m_z^2 & 1 \end{vmatrix}$$

This function depends only on l and m and not on the choice of the four points. Geometrically, a line m with $sign(l, m) > 0$ is tangent to some helix turning clockwise around l as it moves in the direction of l, and m with $sign(l, m) > 0$ is tangent to a counterclockwise helix.

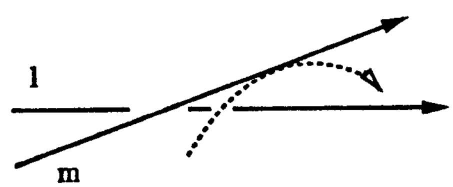

Figure 1: $side(l, m) > 0$

$sign(l, m) = 0$ when m and l intersect, or when m and l are parallel. A fixed line l divides linespace into three equivalence classes, where $sign(l, m)$ has the same value for all m in any class.

A continuous mapping of the closed interval $[0, 1]$ to linespace is a *path*. We can think of any path in linespace as a motion of a line in E^3, from a position m_0 to another position m_1. If, for some directed line l, $side(l, m_1) \geq 0$, and $side(l, m_0) \leq 0$, there is no motion which takes m_0 into m_1 without either intersecting l or becoming parallel to l. This immediately gives us

LEMMA 3.1. *Any path in linespace satisfies the Crossing Condition.*

A set f in linespace is *connected* if between any two points in f there is a finite-length path, completely contained in f. For example, for a fixed line l, the equivalence class $\{m : side(l,m) > 0\}$ is connected because you can move any line in the set into any other, without intersecting l.

A set L of n lines induces a partition of linespace into equivalence classes, where, for each $l \in L$, $sign(l,m)$ has the same value for every m in any class. Usually these equivalence class are open sets. The closure of such an equivalence class is a *face*, by analogy with the faces in an arrangement. The faces in linespace induced by a general set of lines need not be connected [CEGS 89], [Pel 90].

For a set B of directed lines, $C(B)$ is the set of lines which intersect, or are parallel to, all the lines in B. For example, $C(\{l\})$ is the three-dimensional set of lines which intersect or are parallel to l.

4 Parameterization and Octants

Any line in E^3 can be represented parametrically by $u + tv$, where u is a point on the line, v is a direction vector, and t is a scalar parameter. If we give up the ability to represent lines normal to the z axis, we can normalize the vectors, so that $v = (x + x', y + y', 1)$, and $u = (x, y, 0)$, where x' and y' are the slopes dx/dz and dy/dz. Thus the four parameters (x', y', x, y) can represent almost any directed line m. We use the notation $x'(m)$ to represent the x' parameter of line m, and so on.

We divide linespace into octants based on the sign of x' and y'. Consider, without loss of generality, the positive octant.

$$\hat{O} = \{m : x'(m) > 0 \text{ and } y'(m) > 0\}$$

$\hat{O}$ is an open set. In order to be able to define the minimum of the octant, we consider instead a compact set O defined using an arbitrarily small symbolic constant ϵ. For any $m \in O$,

$$\begin{aligned} \epsilon &\le x'(m) \le 1/\epsilon \\ \epsilon &\le y'(m) \le 1/\epsilon \\ -1/\epsilon &\le x(m) \le 1/\epsilon \\ -1/\epsilon &\le y(m) \le 1/\epsilon \end{aligned}$$

Notice that the lines in the set $\overline{O} = \{m : x'(m) = 0 \text{ or } y'(m) = 0\}$ belong to no octant.

We assign a lexicographic ordering to the parameters, which imposes a total ordering on the set of directed lines. We test whether two lines have the relation $m_0 < m_1$ by comparing $x'(m_0) < x'(m_1)$, and, if they are the same, $x'(m_0) < x'(m_1)$, and so on.

For O or any compact subset of O, the most significant parameter x' is a continuous function on the set, y' is a continuous function on the compact subset consisting of the points with minimal x', and so on. Since there is a total ordering on the points, the set has a unique minimum.

We define the subroutine $min(B)$ to return the minimum $m \in C(B) \cap O$. This is theoretically computable in constant time; it can also be implemented efficiently. The ϵ constraints can be interpreted as directed axial lines, forming a bounding box around O. We can avoid having to intersect $C(B)$ with a bounding box, however, by calling different subroutines based on the number of lines in B parallel to each axis. Intuitively, we can find in each case $m = min(B)$ by considering the minimum such m as x' and y' go to zero.

5 Line Transversal of Boxes and Rectangles

In this section, we reduce the problem of finding a line transversal of a set of axial boxes and rectangles to finding a feasible line in a linespace arrangement induced by axial lines, using ideas from [Pel 90] and [HT91]. The overall approach is to search the set $\overline{O}$ of lines on the boundary of the octants, and then each octant in turn for a line transversal.

There is a simple $O(n)$ algorithm to search $\overline{O}$. For each axis a in turn, we search for a line transversal normal to a. First we find a plane P normal to a which intersects all of the objects, if one exists, by examining each object and keeping track of the interval on a which might intersect P. If we succeed, we try to find a line transversal of the traces of the objects in P. This two dimensional version of the problem can be formulated as linear programming problem with two variables and $O(n)$ constraints. So this simple algorithm runs in $O(n)$ time, and illustrates that the set of line transversals normal to each axis a is connected.

The rest of the paper is devoted to the algorithm to search a particular octant for a line transversal. We consider without loss of generality the positive octant O.

For any axial rectangle AR, consider picking any arbitrary $m \in O$ intersecting AR. We direct the lines supporting the edges of AR so that they form a cycle, and m is feasible with respect to all of them. All the lines of O which intersect the rectangle are exactly those which are feasible with respect to all four of the directed axial lines.

Now consider the projection of an axial box AB onto the plane $z = 0$ along an arbitrary line $m \in O$. For any m, the boundary of the projected figure is formed by the projections of the same six lines. Again we direct these six lines so that they form a cycle and a line $m \in O$ intersects AB if and only if m is feasible with respect to all the directed axial lines.

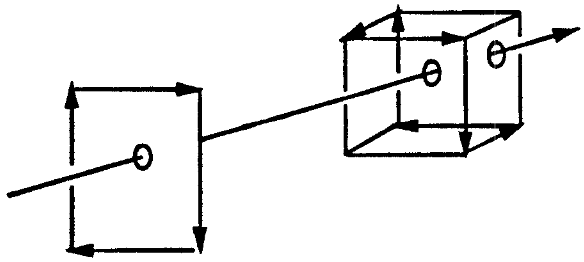

Figure 2: reduction to directed lines problem

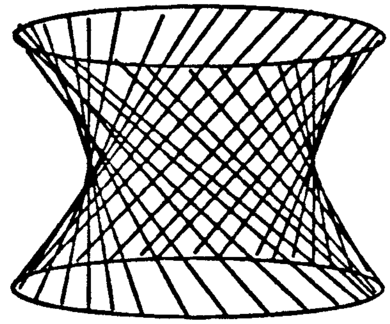

Figure 3: a hyperboloid of lines

Thus we reduce the problem of finding a line transversal of a set of axial boxes and rectangles to eight instances of the following problem:

PROBLEM 1: Given a set of axial lines L, find a line $m \in O$ which is feasible with respect to every $l \in L$.

6 Canonical Paths and Connectivity

Now we show that the sets O and L in this problem satisfy conditions 1 and 2, which means that it can be solved by the *GLP* algorithm.

We have already seen that any path in linespace satisfies the Crossing Condition. To establish the Convexity Condition and the Endpoint Condition, we define a canonical path $path(m_0, m_1)$ between any $m_0, m_1 \in O$. There are two kinds of canonical paths.

If m_0 and m_1 are contained in a common plane P, then they intersect at some point $p \in P$ (when m_0 and m_1 are parallel, we can think of p as a point at infinity). Let N be the pencil of lines in P through p. N intersects O in a single connected component, which contains both m_0 and m_1. We define $path(m_0, m_1)$ as the closed segment of that pencil bounded by m_0 and m_1. This is a ***planar canonical path***.

If m_0 and m_1 do not lie in a common plane, we can construct three non-intersecting axial lines $\{a, b, c\}$ through them, where a, b and c are parallel to the x, y and z axes, respectively. The set $H = C(\{a, b, c\})$ of lines through a, b and c is one set of ruling lines on a hyperboloid of one sheet, which we shall call a ***hyperboloid of lines***.

The intersection of this hyperboloid of lines with the positive octant O, that is, the part of it where both slopes x' and y' are positive, is a connected set of lines. To make all this more credible, we define H explicitly.

Let $a = (x, a_y, a_z, 1)$ be the line parallel to the x axis with $y = a_y$ and $z = a_z$. A line $(x', y', x, y) \in C(\{a\})$ if

$$\begin{vmatrix} 0 & a_y & a_z & 1 \\ 1 & a_y & a_z & 1 \\ x & y & 0 & 1 \\ x + x' & y + y' & 1 & 1 \end{vmatrix} = 0$$

Evaluating this determinant gives us an equation with coefficients dependent on the constants a_z and a_y. We get similar equations from the lines $b = (b_x, y, b_z, 1)$ and $c = (c_x, c_y, z, 1)$, producing a system of three equations in the four unknowns (x', y', x, y).

$$\begin{aligned} a_y + a_z(y') - y &= 0 \\ b_x - b_z(x') + x &= 0 \\ c_x(y') + c_y(y') + (xy' - yx') &= 0 \end{aligned}$$

We can solve this system to express y',x,and y in terms of x'. We find that the two direction parameters are related by

$$y' = \frac{(a_y - c_y)x'}{(b_x - c_x) + (b_z - a_z)x'} \tag{6.1}$$

For any $m \in H \cap O$, $x'(m)$ and $y'(m)$ must satisfy this equation. For convenience, we write this

$$y' = \frac{c_1 x'}{c_2 + x'} \tag{6.2}$$

Clearly y' is defined for any x', so without loss of generality, let $m \in H$ with $x'(m_1) \geq x'(m) \geq x'(m_0)$. At both m_0 and m_1, the parameter $y' \geq \epsilon$, so the sign of $c_1 x'$ must be equal to the sign of $c_2 + x'$. If $c_1 > 0$, then they are both positive at both m_0 and m_1. Since $c_1 x'$ and $c_2 + x'$ are linear, they must also both be positive at any m inbetween m_0 and m_1. So $x'(m) > 0, y'(m) > 0$, and $m \in O$.

If $c_1 < 0$, a symmetrical argument shows again that any $m \in O$. Thus $H \cap O$ is connected. We define the canonical path $path(m_0, m_1)$ as the segment of H bounded by m_0 and m_1. This is a ***hyperbolic canonical path***.

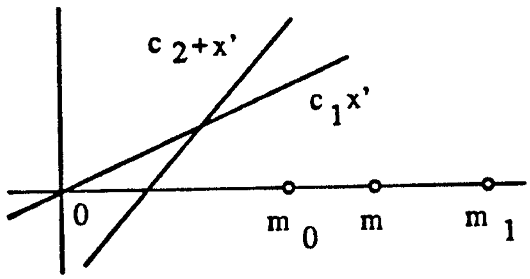

Figure 4: m has to be in O

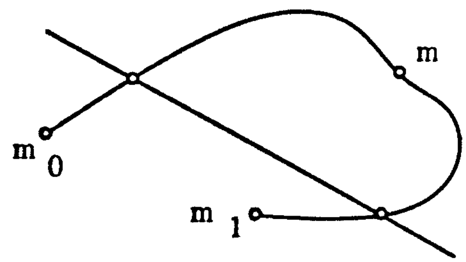

Figure 5: this can't happen

LEMMA 6.1. *All canonical paths in O satisfy the Convexity Condition with respect to a set L of axial lines.*

Proof. Recall that the Convexity Condition states that for any $m_0, m_1 \in O$, and any axial line l, either $path(m_0, m_1) \subseteq C(\{l\})$ or $path(m_0, m_1)$ intersects $C(\{l\})$ in at most a single line.

Either $path(m_0, m_1)$ is planar or hyperbolic. In the first case, an axial line l either lies in the plane P containing the pencil N, or intersects it in a point p, or it fails to intersect it at all. If l lies in P,, every $m \in N$ is also in $C(\{l\})$. If p is the common point of the lines of N, again every $m \in N$ is also in $C(\{l\})$. Otherwise only the single line $m \in N$ containing the point p is in $C(\{l\})$, and m may or may not be in $path(m_0, m_1)$. If l fails to intersect P, then $N \cap C(\{l\})$ is empty.

In the second case $path(m_0, m_1)$ is a segment of the hyperboloid of lines $H = C(\{a, b, c\})$. Assume, without loss of generality, that l is parallel to a, so that they lie in some common plane P. The lines b and c intersect P, and, since b and c are disjoint, they do so in two distinct points. So there is one line $m \subseteq P$ which is both in H and in $C(\{l\})$, and again m may or may not be in $path(m_0, m_1)$.

In any case, there is at most one line $m \in path(m_0, m_1) \cap C(\{l\})$.

This leads to

THEOREM 6.1. *Every face in O induced by a set of axial lines is connected.*

Proof. A face f is connected if there is some path, completely contained in f, between any two points $m_0, m_1 \in f$. We establish that $path(m_0, m_1)$ is such a path. If there were some point $m \in path(m_0, m_1)$ such that $m \notin f$, then for some axial line l, $side(l, m) < 0$ while $side(l, m_0) \geq 0$ and $side(l, m_1) \geq 0$. But then $path(m_0, m_1) \cap C(\{l\})$ must consist of at least two lines, which is impossible. So $path(m_0, m_1) \subseteq f$ and f is connected.

The set of feasible lines $m \in O$ is a face induced by the set L of axial lines, so it is connected. Since there are only eight octants, and the set of line transversals in the set $\overline{O}$ has a constant number of connected components, we have:

COROLLARY 6.1. *The set of directed line transversals of axial objects has a constant number of connected components.*

Theorem 6.1 can also be interpreted as the solution of a three-dimensional motion planning problem.

PROBLEM 2: *Given a set of obstacles consisting of n axial lines in O, and two lines $m_0, m_1 \in O$, compute a collision-free motion, if one exists, from m_0 to m_1.*

In $O(n)$ time we can determine if m_0 and m_1 are in the same face. If so, the canonical path $path(m_0, m_1)$ defines a collision-free motion, and can be computed in constant time. If not, no such motion exists.

7 Remaining Conditions

LEMMA 7.1. *All canonical paths in O satisfy the Endpoint Condition.*

Proof. Recall that the Endpoint Condition states that for any canonical path $path(m_0, m_1)$, with $m_1 > m_0$, if $m \in path(m_0, m_1)$, then $m_1 \geq m \geq m_0$.

First we show that every canonical path either has the same value everywhere for x' or has at most one line with any particular x'. If $path(m_0, m_1)$ is planar, either $x'(m)$ is the same for every $m \in path(m_0, m_1)$, or, for any β, there is at most one $m \in path(m_0, m_1)$ such that $x'(m) = \beta$. On a hyperbolic path, each y' determines a unique x' in O, so for any β, there is at most one $m \in path(m_0, m_1)$ such that $x'(m) = \beta$. We can make a similar argument for the other parameters.

Now assume, for the purpose of contradiction, that $path(m_0, m_1)$ does not satisfy the Endpoint Condition. Since x' is the most significant parameter, either x' is constant along $path(m_0, m_1)$, or there is some $m \in path(m_0, m_1)$, such that $x'(m) > x'(m_1) > x'(m_0)$. But x' is continuous along $path(m_0, m_1)$, so there must be another point m', in between m and m_0, with $x'(m') = x'(m_1)$, which is impossible.

It remains possible that x' is constant along $path(m_0, m_1)$. But since all the parameters cannot be constant, every $path(m_0, m_1)$ must satisfy the Endpoint Condition.

Finally, we need to establish that the sets O and L satisfy Condition 2.

LEMMA 7.2. *For any $A \subseteq L$, if $min(A)$ exists, then there are at most 4 axial lines $l \in A$ such that $min(A - \{l\}) < min(A)$.*

Proof. Let $B = \{l : min(A - \{l\}) < min(A)\}$. If $|B| < 4$, the lemma is trivially true. Assume $|B| \geq 4$, and consider any $B' \subseteq B$ with $|B'| = 4$. At least two $l \in B'$ have to be parallel to the same axis, by the pigeon-hole principal. These two define a plane P which must contain m. The other two must intersect P in two distinct points, otherwise one could be removed from B' without changing $min(B')$. These four define a unique line m, such that $C(B') \cap O = \{m\}$, so $min(B') = m$. Any other line $l \in A$ must be on m, otherwise $C(A) \cap O = \{\}$ and $min(A)$ is undefined. So $min(A) = m$. The removal of any $l \in A$ such that $l \notin B'$ does not change the minimum, and $B = B'$.

This concludes the proof that the sets O and L satisfy conditions 1 and 2. This means that a line $m \in O$ feasible with respect to a set L of directed axial lines can be found by the Generalized Linear Programming algorithm, which in turn gives an $O(n)$ expected time algorithm to find a line transversal for a set of axial boxes or rectangles.

8 Further Work

Recently, Nimrod Megiddo has extended this result by giving a reduction of the problem of finding a line transversal of axial boxes in any dimension to a constant number of small dimensional linear programming problems, which gives a deterministic $O(n)$ algorithm [M91].

Another possible extension would be to finding a line transversal for a set of polytopes with a constant number of edge directions. Unfortunately, we can construct a set of such polytopes which have a set of line transversals with a linear number of connected components.

9 Acknowledgments

The author thanks Seth Teller, Michael Hohmeyer, Ken Clarkson and especially Peter Shor and Raimund Seidel for useful and pleasant discussions. She also thanks Ronitt Rubinfeld for good advice, and Amie Wilkinson for being a topology oracle.

References

[AW87] David Avis and Rephael Wenger. Algorithms for Line Transversals in Space, *Proceedings of the 3d Annual Symposium on Computational Geometry*, pages 300-307, 1987.

[CEGS 89] Bernard Chazelle, Herbert Edelsbrunner, Leonidas Guibas and Micha Sharir. Lines in Space - Combinatorics, Algorithms and Applications, *Proceedings of the 21st Symposium on the Theory of Computing*, pages 382-393, 1989.

[HT91] Michael Hohmeyer and Seth Teller. *Stabbing Isothetic Boxes and Rectangles in $O(n \lg n)$ Time*, Technical Report Number 91/634, University of California at Berkeley, 1991.

[MO88] Michael McKenna and Joseph O'Rourke. Arrangements of lines in 3-space: A Data Structure with Applications, *Proceedings of the 4th Annual Symposium on Computational Geometry*, pages 371-380, 1988.

[M91] Nimrod Megiddo. *personal communication, 1991*

[PS90] Marco Pellegrini and Peter Shor. Finding Stabbing Lines in 3-Dimensional Space, *Proceedings of the 2nd Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 24-31, 1991.

[Pel 90] Marco Pellegrini. Stabbing and Ray Shooting in Three Dimensional Space, *Proceedings of the 6th Annual Symposium on Computational Geometry*, pages 177-186, 1990.

[Pon 91] Carl Ponder. A Search Approach to general Stabbing Problems, *Proceedings of the 3d Canadian Conference on Computational Geometry*, pages 195-202, 1991.

[RS 90] Raimund Seidel. Linear Programming and Convex Hulls Made Easy, *Proceedings of the 6th Annual Symposium on Computational Geometry*, pages 211-215, 1990.

[Tel 91] Seth Teller and Carlo Sequin. Visibility Preprocessing for Interactive Walkthroughs, Computer Graphics, Volume 25, Number 4, SIGGRAPH '91 Conference Proceedings, pages 61-69, 1991.