Computational Topology: Ambient Isotopic Approximation of 2-Manifolds

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Abstract

A fundamental issue in theoretical computer science is that of establishing unambiguous formal criteria for algorithmic output. This paper does so within the domain of computer-aided geometric modeling. For practical geometric modeling algorithms, it is often desirable to create piecewise linear approximations to compact manifolds embedded in $\mathbb{R}^3$, and it is usually desirable for these two representations to be "topologically equivalent". Though this has traditionally been taken to mean that the two representations are homeomorphic, such a notion of equivalence suffers from a variety of technical and philosophical difficulties; we adopt the stronger notion of ambient isotopy. It is shown here, that for any C^2, compact, 2-manifold without boundary, which is embedded in $\mathbb{R}^3$, there exists a piecewise linear ambient isotopic approximation. Furthermore, this isotopy has compact support, with specific bounds upon the size of this compact neighborhood. These bounds may be useful for practical application in computer graphics and engineering design simulations. The proof given relies upon properties of the medial axis, which is explained in this paper.

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1 Topology in Geometric Modeling

Topology has made fundamental contributions to computer science [31]; with applications to domain-theoretic foundations for programming languages [34], an essential role in digital topology [26], and many applications to complexity lower bounds [8,25,39]. Topology also offers direct means for classifying solids, surfaces, and curves; the principal elements of computer-aided geometric modeling and computer graphics. As approximation is an unavoidable aspect of computation with such objects, the question of whether an approximation is "good enough" to preserve the essential features of the object is of central importance. We discuss the concept of ambient isotopy, a topological notion of equivalence, and give conditions sufficient for an approximation to offer such equivalence.

Computer-aided geometric modeling is the process of creating electronic representations of geometric objects, usually in three dimensions. Although curved models can be designed with splines, algebraic surfaces or implicitly via a subdivision process; operations on the models, such as visualization or finite element analysis, might require a piecewise linear (PL) approximation to the model. This work addresses the question of when an object and its approximation should be considered to be topologically equivalent and presents a new theorem which guarantees that such topologically faithful approximation can be achieved on compact, 2-manifolds, which are embedded in $\mathbb{R}^3$ and have continuous second derivatives. Recent discussions have articulated the notion of 'computational topology' [14], primarily as a combination of topology and computational geometry. Most of the focus to date [14,17] has ignored differentiability and approximation. To the contrary, this work emphasizes the integration of computational topology, differential topology and approximation.

2 Ambient Isotopy and Approximation

Many geometric approximation algorithms offer no guarantees about the topology of the output. Sometimes it is guaranteed that the output is homeomorphic to a desired manifold [2,24]. We argue here that a guarantee of homeomorphism is insufficient for many of the applications for which the algorithms are designed. Rather, examples are given for preferring a stronger equivalence relation based upon ambient isotopy.

Definition 1 *If X and Y are subspaces of $\mathbb{R}^3$, then X and Y are ambient isotopic if there is a continuous mapping*

$$F : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$$

such that for each $t \in [0,1]$, $F(\cdot,t)$ *is a homeomorphism from* $\mathbb{R}^3$ *onto* $\mathbb{R}^3$ *such that* $F(\cdot,0)$ *is the identity and* $F(X,1) = Y$.

For other fundamental terms, the reader is referred to the text [22].

Although any two simple closed planar curves are ambient isotopic, Figure 1 shows two simple homeomorphic curves, which are not ambient isotopic [4] , where the PL curve is an approximation of the curve on the left. In the right half of Figure 1 the z co-ordinates of some vertices are specifically indicated to emphasize the knot crossings in $\mathbb{R}^3$ (All other end points have $z = 0$). All end points of the line segments in the approximation are also points on the original curve. Having this knotted curve as an approximant to the original unknot would be undesirable in many circumstances, such as graphics and engineering simulations [6,7]. These pathologies can be prevented by extending the topological foundations for geometric modeling to stipulate ambient isotopy for topological equivalence.

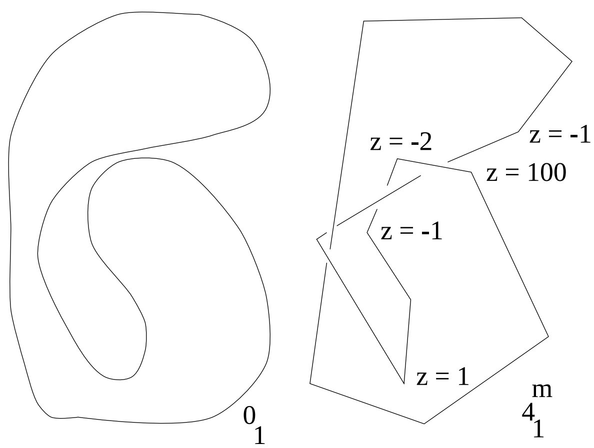

Fig. 1. Nonequivalent Knots

Other problems arise for surfaces (2-manifolds) in three dimensions. Some algorithms compute a triangulated surface C to approximate the boundary F of a closed, finite volume, with a guarantee that C is homeomorphic to F [3,4]. It is well-known that this does *not* guarantee that the complement of C, $\mathbb{R}^3 - C$, is homeomorphic to the complement of F, $\mathbb{R}^3 - F$, meaning that there is no guarantee that F and C are equivalently embedded in $\mathbb{R}^3$. An ambient isotopy between C and F, on the other hand, provides such a guarantee.

The class of PL surfaces presents another domain in which topological guarantees are desirable. Even guaranteeing that the common *edge contraction* operator

[4] The different knot classifications of 0_1 and 4_1^m are indicated.

produces an object homeomorphic to its input requires some care for simplicial complexes [18]. Preservation of genus during approximation by a polygonal mesh [16] also requires considerable care.

The theorem presented here proves an approximation technique which preserves ambient isotopy over an important sub-class of 2-manifolds.

3 Related Work

Throughout the article, for any integer $n \geq 0$, the notation C^n will refer to a function (or manifold) having continuous derivatives of order n; C^∞ will likewise indicate continuous derivatives for all non-negative integers. The related work comes from two broad areas, mathematics and computer science.

3.1 Mathematics

The sub-disciplines of differential topology, PL topology and knot theory are the most relevant, for which key summary references are given, below.

In differential topology, extension of isotopies to ambient isotopies is accomplished on a C^∞ manifold without boundary by constructing a tubular neighborhood [22]. The assumption of C^∞ is natural within that context, but is not invoked here. Rather, our results only require C^2 continuity.

From PL topology, there are necessary and sufficient conditions for an isotopy of compact polyhedra [32, Theorem 4.24, p. 58] to be extended to an ambient isotopy. A common technical tool for proving compact support [5] of an ambient isotopy is the class of functions know as ‘pushes’ [9]. A generalization of a push is used in the proof given here.

In knot theory, ambient isotopy theorems focus upon knot diagrams [20,21]. Finite sequences of Reidemeister moves are known to preserve ambient isotopy over knot diagrams. An alternative definition of ambient isotopy can be given [32] in terms of commutative diagrams.

[5] A function f from X onto itself has compact support if there exists a compact set $A \subset X$ such that f is the identity except possibly on A.

3.2 Computer Science

The catalyst for this paper was work on the problem of constructing an approximating surface mesh given only a sample of points from the surface. This problem was formalized and brought to the attention of the computer graphics community in a seminal 1992 paper [23]. Amenta and Bern [1,2] described the *crust* algorithm for which they could show, under some conditions on the surface and the sample, that the output approximates, geometrically, the surface from which the samples were drawn. A later simplification [3] of this algorithm was shown to produce a PL (triangulated) manifold homeomorphic to the surface from which the samples were taken, using a somewhat complicated argument involving covering spaces. Both of these approaches use the *medial axis*.

Between any two points, $x, y \in \mathbb{R}^3$, let $d(x, y)$ denote the usual Euclidean distance and for any two sets $X, Y \subset \mathbb{R}^3$, let $d(X, Y) = \inf\{d(x, y)|x \in X, y \in Y\}$.

Definition 2 *Let $x \in \mathbb{R}^n$ and $S \subset \mathbb{R}^n$. A point $s \in S$ is a* nearest point on S *to* x *if*

$$d(x, s) = \inf\{d(x, t) \mid t \in S\}.$$

The medial axis *of S, $MA(S)$, is the closure of the set of all points that have at least two distinct nearest points on S.*

This concept was originally defined for object recognition in the life sciences [10]. One investigation of the mathematical properties of the medial axis and its associated transform function [15] is restricted to geometry within the plane. More generally, there has been broad attention to the medial axis in $\mathbb{R}^n$ within the computer science literature, where the topological and differentiable investigations [35,37,38] are most directly relevant to our main theorem about ambient isotopy.

The issue of rigorous proofs for the preservation of topological form in geometric modeling appears to have been first raised regarding tolerances in engineering design [11,12,36], but these papers did not specifically propose ambient isotopy as a criterion. The class of geometric objects considered was appreciably expanded by theorems for ambient isotopic perturbations of PL simplexes and splines [5–7].

In response to the example of Figure 1, a theorem was published for ambient isotopic PL approximations of 1-manifolds [27]. The proof utilizes ‘pipe surfaces’ from classical differential geometry [29]. The improved approximation is shown in Figure 2. There is a related study of curves, comparing them to α-shapes [19] via ambient isotopies [33].

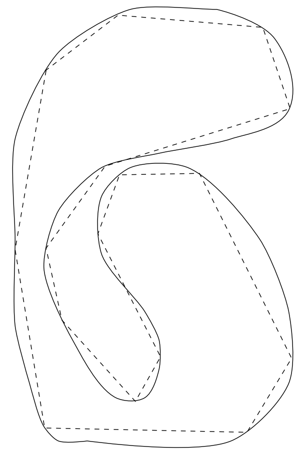

Fig. 2. Ambient Isotopic Approximation

4 Existence of Ambient Isotopy Via the Medial Axis

This section contains the main result of the paper. For the reader's convenience, we briefly summarize the salient aspects of the simplified crust algorithm [3] as the basis for our extensions.

Let M be a C^2, compact, 2-manifold without boundary, with an embedding $E : M \to \mathbb{R}^3$; let F denote the image of M under E. (By *embedding*, we mean that *(i.)* E is C^2, *(ii.)* E is injective, *(iii.)* the differential of E is everywhere non-singular, and *(iv.)* that M is homeomorphic to F when F is given the relative topology inherited from $\mathbb{R}^3$. Our terminology follows [13, §II]; see Appendix A for more discussion.)

It was shown [3] that there exists a PL (triangulated) manifold C which is embedded in $\mathbb{R}^3$ such that every vertex of C belongs to F and that there also exists a homeomorphism $h : C \to F$, defined by mapping each point of C to the nearest point on F. Furthermore, for each $c \in C$,

$$d(c, h(c)) < 0.165 \cdot d(h(c), MA(F)).$$

The creation of such a homeomorphism assumes that the vertices of C form a 'sufficiently dense' sampling of F, meaning that for every point x on F the distance from x to the nearest vertex of C is at most some 0.08 times the distance from x to $MA(F)$ [3]. The particular values of 0.165 and 0.08 were chosen for convenient numerical manipulation within the cited proof, and could possibly be improved.

Lemma 3 has appeared in the literature [37], using different terminology. In order to keep this paper self-contained, we also present its proof in Appendix A. Note that Lemmas 3, 4 and 7 would be false without some hypothesis about the differentiability of F, where our assumption of C^2 is sufficient.

Lemma 3 $d(F, MA(F)) > 0$.

PROOF. See Appendix A. □

Lemma 4 *If q is any point not on F, then any line segment from F to q having length $d(q, F)$ is perpendicular to F.*

PROOF. See, e.g., [30, Theorem 1]. □

The following definition, restated from [38], is useful within our next lemma.

Definition 5 *Let A be a closed subset of $\mathbb{R}^n$. A point $p \in \mathbb{R}^n$ is a* non-extender *relative to A if there exists a shortest path from A to p which cannot be extended to be a shortest path to any point beyond p.*

Proposition 6 *Let p be a non-extender relative to F, then $p \in MA(F)$.*

PROOF. See [38], where it is shown that $MA(F)$ is in fact the closure of the set of non-extenders relative to F. (N.b., Wolter adopts different terminology for these sets: his *cut locus* is equivalent to our *medial axis*. He shows that the closure of the set of non-extenders is the same as the closure of the set of points with at least two nearest neighbors, as in Definition 2). See [38, Definition 3.4.I], [38, Definition 3.4.II], and the following remarks. [6] □

Lemma 7 *Let $\vec{n}_1$ and $\vec{n}_2$ be unit length normals to F at two distinct points p_1 and p_2. Suppose that the two lines $\ell_1(t) = p_1 + t\vec{n}_1$ and $\ell_2(s) = p_2 + s\vec{n}_2$ intersect at a point q, where $t, s \in [0, \infty)$. Then*

$$\max(d(p_1, q), d(p_2, q)) \geq d(F, MA(F)).$$

[6] Note, finally, that Wolter also defines a notion of medial axis, only directly applicable to compact, regular closed subsets of $\mathbb{R}^n$.

PROOF. Let $\ell_1(t_q) = \ell_2(s_q) = q$; without loss of generality, we assume that $t_q = d(p_1, q) > s_q = d(p_2, q) > 0$. For $t \in [0, \infty)$, let $\overline{B}_t$ be the closed ball centered at $\ell_1(t)$ of radius t and note that for all $t \in [0, \infty)$, we have that $p_1 \in \partial\overline{B}_t$ (where the notation ∂ indicates the boundary). By Proposition 10 in Appendix A, there exists some $t_0 > 0$ such that for all $t \in [0, t_0]$, $|\overline{B}_t \cap F| = 1$. Let

$$t_1 = \sup\{t \in [0, \infty) \mid |\overline{B}_t \cap F| = 1\};$$

note that $\{p_1, p_2\} \subset \overline{B}_{t_q}$ so that $t_1 \leq t_q$. Any point of $\overline{B}_{t_1} \cap F$ must be on the boundary of $\overline{B}_{t_1}$ because any interior point of $\overline{B}_{t_1}$ which also appeared in F would also be contained in $\overline{B}_{t_1 - \epsilon}$ for some appropriately small positive ϵ; this is precluded by our choice of t_1. Likewise, observe that for any $t > t_1$, there are points of F in the interior of $\overline{B}_t$ and hence that $\ell_1(t_1)$ is a non-extender. By Proposition 6 we must have $\ell(t_1) \in MA(F)$. Hence $\max(d(p_1, q), d(p_2, q)) \geq d(p_1, \ell_1(t_1)) \geq d(F, MA(F))$, as desired. $\square$

4.2 Main Theorem

In this section, we show that there exists an ambient isotopy of compact support between the surfaces F and C. This is the main theorem of this paper. To establish this, we first define the following functions.

Consider the vector from c to its nearest point, $h(c)$ on F. Let c^* denote this nearest point to c and let $\overrightarrow{cc^*}$ denote the vector $c^* - c$. Any point of C such that $c = c^*$ will be called a *fixed point*[7] of C. Let $\dot{C}$ denote the set of all fixed points of C. Note, by Lemma 4, that for any $c \notin \dot{C}$, the vector $\overrightarrow{cc^*}$ is normal to F at c^*.

Define the two functions $\hat{h}$ and $\check{h}$, from C into $\mathbb{R}^3$ so that

$$\begin{aligned}\hat{h}(c) =& c + 2 \cdot \overrightarrow{cc^*}, \text{ and}\\ \check{h}(c) =& c - \overrightarrow{cc^*}.\end{aligned}$$

Let $\hat{F} = image(\hat{h})$ and $\check{F} = image(\check{h})$.

Lemma 8 *The functions $\hat{h} : C \to \hat{F}$ and $\check{h} : C \to \check{F}$ are homeomorphisms.*

[7] Each sample point is a fixed point of C and there is nothing that precludes a subset of some triangle of C from being a subset of F, meaning that all points of that subset would also be fixed points.

PROOF. Observe that for any $c \in C$,

$$d(c^*, c) = d(c^*, \hat{h}(c)) < .165 \cdot d(c^*, MA(F)); \text{and}$$
$$2 \cdot d(c^*, c) = d(c^*, \check{h}(c)) < .33 \cdot d(c^*, MA(F)).$$

Since both $\hat{h}(c)$ and $\check{h}(c)$ lie along normals to F at c^*, these indicated small distances from $MA(F)$ are sufficient to conclude from Lemma 7 that both $\hat{h}$ and $\check{h}$ are 1-1. The other properties required for $\hat{h}$ and $\check{h}$ to be homeomorphisms follow easily. $\square$

Theorem 9 *There exists an ambient isotopy of compact support between the surfaces F and C.*

PROOF. Define the function $\Lambda : C \times [0,1] \to \mathbb{R}^3$ so that

$$\Lambda(c,t) = \begin{cases} \check{c} + 3t\overrightarrow{\check{c}c^*}, & \text{if } t \in [0,1/3], \text{ where } \check{c} = \check{h}(c), \\ c + 3(t-1/3)\overrightarrow{cc^*}, & \text{if } t \in [1/3,2/3], \text{ and} \\ c^* + 3(t-2/3)\overrightarrow{cc^*} & \text{if } t \in [2/3,1]. \end{cases}$$

Note that between 0 and $1/3$, this deforms the surface $\check{F}$ into C, between $1/3$ and $2/3$ this deforms[8] the surface C into F, and between $2/3$ and 1, this deforms the surface F into $\hat{F}$. Let $i(\check{F}, \hat{F})$ denote the set $\{\Lambda(c,t) \mid c \in C, t \in [0,1]\}$. By Lemma 7, Λ is 1-1 on the set $(C \setminus \dot{C}) \times [0,1]$. (On $c \in \dot{C}$, $\Lambda(c,s) = \Lambda(c,t)$ for all $s, t \in [0,1]$.)

Now, let $\mathrm{id} : [0,1] \to [0,1]$ be the identity map and let $r : [0,1] \to [0,1]$ be the function

$$r(x) = \begin{cases} 2x, & \text{if } x \in [0,1/3], \text{ and} \\ (1+x)/2 & \text{if } x \in [1/3,1]. \end{cases}$$

Let $\Lambda^r(c,s) = \Lambda(c, r(s))$. Observe that $\Lambda^r(c,t)$ also deforms $\check{F}$ into $\hat{F}$, but the set $C \times \{1/3\}$, which was mapped to C under Λ is now mapped to $F = \Lambda(C, 2/3) = \Lambda^r(C, r(1/3))$. We interpolate between $\Lambda(c,s)$ and $\Lambda^r(c,s)$ to define the ambient isotopy.

Define the function

$$G : \mathbb{R}^3 \times [0,1] \to \mathbb{R}^3$$

so that for each $p \in (\mathbb{R}^3 \setminus i(\check{F}, \hat{F})) \cup \dot{C}$, $G(p,t) = p$ for all t; for a point $p \in i(\check{F}, \hat{F}) \setminus \dot{C}$, let $(c_p, s_p) = \Lambda^{-1}(p)$ and define

$$G(p,t) = \Lambda\left(c_p, [t \cdot r + (1-t) \cdot \mathrm{id}](s_p)\right),$$

[8] The choices of the values $1/3$ and $2/3$ here are arbitrary, but are *not* related to the use of the constant 0.33 within the second inequality in the proof of Lemma 8. The values of $1/3$ and $2/3$ used here could be replaced by any values α, β, such that $0 < \alpha < \beta < 1$. The technical details of the proof would change, but the underlying logic would be the same.

where $[t \cdot r + (1-t) \cdot \mathrm{id}] : [0,1] \to [0,1]$ is the map $x \mapsto t \cdot r(x) + (1-t) \cdot x$.

Observe that, for each t, $[t \cdot r + (1-t) \cdot \mathrm{id}]$ is a bijection on $[0,1]$ and hence the map $G(\cdot, t)$ is bijective for fixed $t \in [0,1]$. To see that $G(\cdot, t)$ is continuous, observe that the boundary of $i(\check{F}, \hat{F})$ is precisely $\hat{F} \cup \check{F}$; as $G(\cdot, t)$ is clearly continuous in the interior of $i(\check{F}, \hat{F})$ it suffices to check that $G(\cdot, t)$ fixes every point in $\hat{F} \cup \check{F}$, which it does since both 0 and 1 are fixed by $[t \cdot r + (1-t) \cdot \mathrm{id}]$. Finally, observe that for all p, $\|G(p, t_1) - G(p, t_2)\| \leq |t_1 - t_2| \cdot .165 \cdot d(F, MA(F))$, from which it easily follows that G is continuous on $\mathbb{R}^3 \times [0,1]$.

Note that $G(p, 0)$ is the identity function on $\mathbb{R}^3$, that $\{G(c, 1) \mid c \in C\} = F$ and that G is an ambient isotopy between C and F. $\square$

5 Conclusions and Future Directions

Theoretical foundations for geometric algorithms should address topological characteristics, as is demonstrated by example in this paper. The main theorem proven in this paper significantly expands the class of manifolds known to have PL approximations which preserve the topological characteristics of the given manifold. Specifically, sufficient conditions are given for a PL (triangulated) approximation to be ambiently isotopic to a given C^2 compact 2-manifold, without boundary, where the manifold is assumed to be embedded in $\mathbb{R}^3$. Furthermore, it is shown that this isotopy has compact support, with a specific upper bound upon the distance of this compact set from the original manifold. This quantitative bound may be useful in practical applications in computer graphics and engineering. The magnitude of this bound is generally inversely proportional to the number of approximating facets. Typically, in practical computing applications, it is desirable to minimize this number of facets. There is no claim in this paper about achieving any such minimality and this minimality relation remains to be investigated, both for the work presented here for 2-manifolds and for work by previous authors on 1-manifolds.

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References

[1] Amenta, N. and Bern, M. *Surface reconstruction by Voronoi filtering*, Discrete and Computational Geometry, 22, 481-504, 1999. An earlier version appeared in the *14th Annual ACM Symposium on Computational Geometry*, pages 39-48, (1998).

[2] Amenta, N., Bern, M. and Kamvysselis, M., *A new Voronoi-based surface reconstruction algorithm*, Proc. ACM SIGGRAPH '98, 1998, 415 - 421.

[3] Amenta, N., Choi, S., Dey, T. and Leekha, N., *A simple algorithm for homeomorphic surface reconstruction*, ACM Symposium on Computational Geometry, 2000, 213-222. Submitted to International Journal of Computational Geometry and its Applications.

[4] Amenta, N., Choi, S. and Kolluri, R., *The power crust, unions of balls, and the medial axis transform*, Computational Geometry: Theory and Applications, 19, 2001, 127-153.

[5] Andersson, L.-E., Dorney, S. M., Peters, T. J. and Stewart, N. F., *Polyhedral perturbations that preserve topological form*, Computer Aided Geometric Design, 12, 1995, 785 - 799.

[6] Andersson, L.-E., Peters, T. J. and Stewart, N. F., *Self-intersection of composite curves and surfaces*, Computer Aided Geometric Design, (15), No. 5, 507 - 527, 1998.

[7] Andersson, L.-E., Peters, T. J. and Stewart, N. F., *Equivalence of topological form for curvilinear geometric objects*, International Journal Computational Geometry and Applications, (10), No. 6, 2000, 609 - 622.

[8] Ben-Or, M., *Lower bounds for algebraic computation trees*, Proc. ACM Symposium on the Theory of Computing, 1983, 80-86.

[9] Bing, R. H. (1983) *The Geometric Topology of 3-Manifolds*, American Mathematical Society Colloquium Publications, Volume 40, Providence, RI: American Mathematical Society.

[10] Blum, H., *Biological shape and visual science (Part I)*, Journal of Theoretical Biology, 38, 1973, 205 - 287.

[11] Boyer, M. and Stewart, N. F., *Modeling spaces for toleranced objects*, International Journal of Robotics Research Vol. 10, No. 5, 1991, 570-582.

[12] Boyer, M. and Stewart, N. F., *Imperfect form tolerancing on manifold objects: a metric approach*, International Journal of Robotics Research, Vol. 11, No. 5, 1992, 482-490.

[13] Bredon, G., Topology and Geometry, Springer-Verlag, New York, 1993.

[14] Emerging Challenges in Computational Topology, report from the NSF-funded Workshop on Computational Topology, Miami Beach, FL, June 11 - 12, 1999.

[15] Choi, H. I., Choi, S. W. and Moon, H. P., *Mathematical theory of the MA transform*, Pacific Journal of Mathematics, 181:1 1997, 57-88.

[16] Cohen, J., Varshney, A., Manocha, D., Turk, G., Weber, H., Agarwal, P., Brooks, F., Jr., and Wright, W., *Simplification envelopes*, Proceedings of ACM SIGGRAPH '96, 119 - 128.

[17] Dey, T. K, Edelsbrunner, H., and Guha, S., *Computational topology*, in Advances in Discrete and Computational Geometry (Contemporary Mathematics 223), Chazelle, B., Goodman, J. E., and Pollack, R., American Mathematical Society, Providence, RI, 1999, 109 - 143.

[18] Dey, T. K., Edelsbrunner, H., Guha, S. and Nekhayev, D., *Topology preserving edge contraction*, Publications de l' Institut Mathematique (Beograd), Vol. 60 (80), 1999. Also, Technical Report RGI-Tech-98-018, Raindrop Geomagic Inc., Research Triangle Park, North Carolina, 1998.

[19] Edelsbrunner, H., and Mücke, *Three-dimensional alpha shapes*, ACM Trans. Graphics 13(1), 1994, 43 - 72.

[20] N. D. Gilbert and T. Porter, Knots and Surfaces, Oxford Science Publications, 1994.

[21] G. Hemion, The Classification of Knots and 3-Dimensional Spaces, Oxford Science Publications, 1992.

[22] Hirsch, M.W., Differential Topology, Springer-Verlag, New York, 1976.

[23] Hoppe, H, DeRose, T., Duchamp, T., McDonald, J., and Stuetzle, W., *Surface reconstruction from unorganized points*, Proc. ACM SIGGRAPH '92, 1992, 71 - 78.

[24] Hu, C.-Y. , Towards Robust Interval Solid Modeling for Curved Objects. PhD thesis, Massachusetts Institute of Technology, Cambridge, MA, May 1995.

[25] Kahn, J., Saks, M. and Sturtevant, *Topological approach to evasiveness*, 24th Annual IEEE Symposium on Foundations of Computer Science, Tucson, Arizona, Nov. 7 - 9, 1983, 31–33.

[26] Kong, T. Y. and Rosenfeld, A., *Digital topology: a comparison of the graph-based and topological approaches*, Reed, G. M., Roscoe, A. W. and Wachter, R. F., (Eds.), Topology and Category Theory in Computer Science, Clarendon Press, Oxford, Great Britain, 1991 273 - 287.

[27] Maekawa, T., Patrikalakis, N. M., Sakkalis, T. and Yu, G., *Analysis and applications of pipe surfaces*, Computer-Aided Geometric Design, 15(5), 1998, 437-458.

[28] Marsden, J. E. and Tromba, A. J., Vector Calculus, 3rd Edition, Freeman

[29] Monge, G., Application de l'Analys à la Geometrie, Bachelier, Paris, 1850.

[30] Pegna, J. and Wolter, F.-E., *Surface curve design by orthogonal projection of space curves onto free-form surfaces*, Journal of Mechanical Design, 118, March, 1996, 45 - 52.

[31] Reed, G. M., Roscoe, A. W. and Wachter, R. F., (Eds.), Topology and Category Theory in Computer Science, Clarendon Press, Oxford, Great Britain, 1991.

[32] Rourke, C. P. and Sanderson, B. J., Introduction to Piecewise-Linear Topology, Springer-Verlag, New York, 1972.

[33] Sakkalis, T. and Charitos, Ch., *Approximating curves via alpha shapes*, Graphical Models and Image Processing, 61, 1999, 165 - 176.

[34] Scott, D. S., *Domains for denotational semantics*, in Neilson, M. and Schmidt, E. M., (Eds.), Automata, Languages and Programming: Proceedings 1982, Springer-Verlag, Berlin, Lecture Notes in Computer Science 140, 1982.

[35] Sherbrooke, E. C., Patrikalakis, N. M., and Wolter, F-E., *Differential and topological properties of medial axis transforms*, Graphical Models and Image Processing, 58, #6, 1996, 574 - 592.

[36] Stewart, N. F. *Sufficient condition for correct topological form in tolerance specification*, Computer-Aided Design 25, No. 1, 1993, 39-48.

[37] Wolter, F.-E., *Cut locus and medial axis in global shape interrogation and representation*, Design Laboratory Memorandum 92 - 2, MIT, Department of Ocean Engineering, Cambridge, MA, September 1993.

[38] Wolter, F.-E., Cut Loci in Bordered and Unbordered Riemannian Manifolds, Dissertation, Vom Fachbereich Mathematik der Technischen Universität Berlin zur Verleihung des akademischen Grades.

[39] Yao, A C.-C., *Decision tree complexity and Betti numbers*, ACM Symposium on Theory of Computing 26, Proceedings of the Twenty-Sixth Annual ACM Symposium on the Theory of Computing, Montréal, Québec, Canada, May 23 - 25, 1994, 615–624.

A Proof of Lemma 3

Recall that an n-dimensional C^k manifold is a (second countable, Hausdorff) topological space M together with a collection of maps (which we call *charts*) such that:

(1) each chart ϕ is a homeomorphism $\phi : U \to U' \subset \mathbb{R}^n$, where U is open in M and U' is open in $\mathbb{R}^n$;
(2) each point $x \in M$ is in the domain of some chart; and
(3) for any pair of charts $\phi : U \to U' \subset \mathbb{R}^n$ and $\psi : V \to V' \subset \mathbb{R}^n$, the "change of coordinates map" $\phi\psi^{-1} : \psi(U \cap V) \to \phi(U \cap V)$ is C^k.

Such a collection of maps is said to be an *atlas* for M. (Note that any atlas can be enlarged so that it is in fact maximal with respect to properties 1, 2, and 3, above.)

Prior to proving Lemma 3, we state and prove the following supporting proposition. Again, similar arguments appear in the literature [35,37].

Proposition 10 *Let p be a point of F and $\vec{n} \in \mathbb{R}^3$ a unit vector normal to F at p. Then there exists some $\epsilon > 0$ such that the closed ball $\overline{B}_\epsilon(p + \epsilon\vec{n})$ of radius ϵ centered at $p + \epsilon\vec{n}$ intersects F at exactly one point, p.*

PROOF. The argument has two parts, dealing with the local and global structure of F, respectively. We begin with the local argument. Consider a point $\tilde{p} \in M$ and its image $p = E(\tilde{p}) \in F$. Let $\phi : U \to U' \subset \mathbb{R}^2$ be a chart with $\tilde{p} \in U$ and consider the C^2 function $f = E \circ \phi^{-1}$ mapping U' to $\mathbb{R}^3$. Let p' be $\phi(\tilde{p})$. We would like to see that there is a neighborhood W of p' and an $s > 0$ so that any sphere of radius s tangent to the surface $f(W)$ meets $f(W)$ at a single point.

As E is an embedding, the derivative of f at p', denoted $[Df(p')]$, has rank two. Since f is C^2, $[Df(x)]$ is a continuous function of x on U', we may select a neighborhood $V \subset U'$ of p' so that $[Df(x)]$ has rank two on all of V. If we then fix a bounded convex neighborhood W_0 of p' and K a compact set satisfying $W_0 \subset K \subset V$ then

(1) for some $B > 0$, all second partial derivatives of f are bounded above in absolute value by $B > 0$ at every point in W_0,
(2) f is a homeomorphism of W_0 onto the image of f, when given its relative topology, and
(3) for some $b > 0$, for all $x \in W_0$ and all $h \in \mathbb{R}^2$, $\|[Df(x)](h)\| \geq b\,\|h\|$.

By applying a first-order Taylor formula (see, e.g., [28]) to f, for any $x, y \in W_0$,

$$f(y) = f(x) + [Df(x)](h) + r(h)$$

where $h = y - x$ and $r = r_x$ is a remainder term (which depends on x). Fixing, for the moment, a specific $x \in W_0$, the kth coordinate of this remainder term may be bounded by

$$\frac{1}{2}\sum_{i,j=1}^{2} \frac{\partial^2 f_k}{\partial x_i \partial x_j}(h^*_{ij}) h_i h_j,$$

for some h^* on the line segment between x and $x + h$. (Recall that f is C^2, so that these partials exist.) In particular, the quantity above can be bounded in absolute value by $B\,\|h\|^2$, in which case $\|r(h)\|^2 \leq \sqrt{3}B\,\|h\|^2$. So let S be a sphere of radius s, tangent to the surface given by f at the point $f(x)$. As S is tangent to $f(W_0)$, the distance from the point $f(x+h) - r(h)$ to the center of the sphere is at least

$$\sqrt{(b\,\|h\|)^2 + s^2},$$

by the definition of b. Observe that if this distance exceeds $s + \|r(h)\|$, then we can be sure that the point $f(x+h)$ does not lie in (or on) this sphere. If $s < b^2/(4\sqrt{3}B)$ and $\|h\| < b/(\sqrt{6}B)$, then

$$\sqrt{(b\,\|h\|)^2 + s^2} \geq s + \sqrt{3}B\,\|h\|^2 \geq s + \|r(h)\|\,,$$

as desired. In particular, we may select a neighborhood $W \subset W_0$ about p' so that for any $x, y \in W$ and any sphere S of radius less than $b^2/(4\sqrt{3}B)$ which is tangent to F at $f(x)$, we can be guaranteed that $f(y)$ falls outside S.

Now we work with the global structure of F. For a point p on F, we let $\tilde{p}$ denote $E^{-1}(p)$. For each point $p \in F$, select a chart $\phi_p : U_p \to U'_p \subset \mathbb{R}^2$ so that $\tilde{p} \in U_p$; the local argument above asserts that there is a neighborhood W_p of $p' = \phi_p(\tilde{p})$ and an $s_p > 0$ so that every sphere of radius less then s_p tangent to $E \circ \phi_p^{-1}(W_p)$ meets $E \circ \phi_p^{-1}(W_p)$ at a single point. (We index the objects ϕ_p, U'_p, etc. by p rather than $\tilde{p}$ or p' to simplify the notation.) Let $C_p = \phi_p^{-1}(W_p)$ and let $\mathcal{C}$ denote the collection $\{C_p | p \in F\}$. Now, as E is an embedding, the topology of M is identical to the relative topology on F (and E is a homeomorphism when F is given this topology). In particular, for every $p \in F$, there is a neighborhood N_p of p so that $N_p \cap F = E(C_p)$. Let γ_p be small enough so that the open ball $B_{\gamma_p}(p)$ of radius γ_p centered at p is contained in N_p. Finally, for each p, define

$$D_p = C_p \cap E^{-1}(F \cap B_{\gamma_p/2}(p))$$

and let $\mathcal{D}$ be the open cover $\{D_p \mid p \in F\}$. Note that for each $p \in F$, we have $\tilde{p} \in D_p$. If S is a sphere of radius less than $\gamma_p/2$ tangent to the surface F at a point $x \in E(D_p)$, then S is a subset B_{γ_p} and hence any point in the intersection of F and S must actually lie on $E(D_p)$. If, furthermore, S has radius less than s_p, then S meets $E(D_p)$ at a single point. So, let $\epsilon = \min(s_p, \gamma_p/2)$. $\square$

The ϵ constructed above can be taken to be a continuous function of p; as F is compact, the above argument yields a single value of $\epsilon > 0$ applicable to the entire manifold, as expressed in the following corollary.

Corollary 11 *There exists $\epsilon > 0$ so that for every $p \in F$ and unit vector $\vec{n}$ which is normal to F at p, $\overline{B}_\epsilon(p + \epsilon\vec{n}) \cap F = \{p\}$.*

The proof of lemma 3 follows immediately from the above:

PROOF of Lemma 3. It is an immediate consequence of the previous lemma that there is $s_F > 0$ so that any sphere tangent to F of radius s_F or less meets F at a single point. Recall that the medial axis of F is the closure of the set of points in $\mathbb{R}^3$ with at least two nearest neighbors on F. Observe that if m is a point on the medial axis of F and $d(m, F) = c$, then for any $\epsilon > 0$ there is a tangent sphere to F of radius $c + \epsilon$ which contains at least two points of F. Hence $d(F, MA(F)) \geq s_F > 0$. $\square$