Handout
Convex hull conflict lists analysis

1 Data structures

We assume there are two data structures in the general-dimensional convex hull algorithm. One stores the convex hull itself. We won't concern ourselves with the details here, but it has to include (implicitly or explicitly) a graph on the polytope facets, where two facets are connected by an edge if they share a ridge. I must be possible to traverse this graph in time O(number of ridges).

The second data structure is the conflict list. This is a bipartite graph, with un-added points on one side, and existing facets on the other. Every un-added point p is connected by an edge to one existing facet f which is visible from p. Unlike the conflict graph described in the book, we maintain exactly one edge. So if p happens to be added next, we can find all of the visible facets from p (the ones destroyed by the insertion of p) by doing a depth-first-search on the facet graph of the polytope, stopping whenever we see a hidden facet.

After the insertion of p, we remove all of these destroyed facets and the edges to them in conflict list. We need to find a replacement facet for each unadded point that is left isolated. To do this, we do a depth-first search on the destroyed facets *visible from* p. Either we find a facet adjacent to the horizon, or not. In the latter case, p is now inside the convex hull and can be removed from the conflict list. In the former case, we check the new facet f adjacent to that ridge. If f is visible from p, we add the new edge to the conflict list. Otherwise, p is now inside the convex hull.

2 What to count

Lets begin by defining some indicator variables. $I_{p,f}$ says that a specific face f of some intermediate convex hull is in conflict with a specific un-added point p. $D_{i,f}$ says that face f is destroyed at insertion i, and $C_{i,f}$ says that f is created at insertion i.

The worst thing that can happen to a point p at insertion i is that that the face f that p is pointing to will be destroyed, and, even worse, to find a new face to point to, p will have to examine every other face that is destroyed by insertion i and in conflict with p as well. So lets make a random variable $X_{p,i}$ which is the number of conflicts involving p destroyed at insertion i. Lets sum these up for one point:

$$X_{p,i} = \sum_f D_{i,f} \times I_{p,f}$$

Summing over all points:

$$X_i = \sum_p \sum_f D_{i,f} \times I_{p,f}$$

Summing over all insertions, we count every face once whether it is eventually destroyed or not:

$$X = \sum_p \sum_f I_{p,f} \geq \sum_{i=1}^{n} X_i$$

This means that the number of vertices that need to be searched by all points, over all insertions, is at most the total number of "conflicts" between a face and a point that occur. The total cost of maintaining the conflicts is $O(X) = O(\sum_p \sum_p I_{p,f})$.

3 From destruction to creation

We just upper-bounded the number of conflicts that were destroyed by the total number of conflicts. But we can count the total conflicts however we like; and it's easier to count them when they're created then when they're destroyed, that is,

$$X = \sum_{j=1}^{n} X_j = \sum_{j=1}^{n} \sum_p \sum_f C_{j,f} \times I_{p,f}$$

4 Two ways of getting the expectation

The reason this is easier is that the one thing we know about any of these variables is that we can bound the expectation of $C_{j,f}$. We don't know a thing about the $D_{j,f}$ or the $I_{p,f}$. But we know, using backwards analysis, that each point in the ith convex hull is equally likely to have been the last inserted, and every face is adjacent to exactly d points (assuming general position), so $\mathrm{E}[C_{j,f}] \leq d/j$ for all f.

$$\mathrm{E}[X_j] \leq \sum_p \sum_f d/j \times I_{p,f} = d/j \sum_p \sum_f I_{p,f}$$

But we're still stuck with those $I_{p,f}$.

We need to find some other fact about $I_{p,f}$ that we can introduce. We notice that $I_{p,f} = 1$ when the insertion of p destroys f. Any p is equally likely to be the next point inserted. So we can express the expected number of vertices destroyed as an average:

$$\mathrm{E}[D_{j+1}] = \frac{1}{n-j} \sum_p \sum_f I_{p,f}$$

Putting this new fact together with what we had before, we get:

$$\mathrm{E}[X_j] \leq d/j \times (n-j) \times \mathrm{E}[D_{j+1}]$$

This is great, since we got rid of the $I_{p,f}$.

5 From destruction to creation, again

Unfortunately we got the D_{j+1} back, but we can get rid of those the same way we did before, by counting faces when they're created rather than when they're destroyed. First we'll exchange the order of E and $\sum$ a number of times, so that the total expectation is expressed as a sum of the indicator variables $D_{j,f}$.

$$\mathrm{E}[X] = \sum_{j=1}^{n} \mathrm{E}[X_j] = \sum_{j=1}^{n} \frac{d(n-j)}{j} \times \mathrm{E}[D_{j+1}] =$$

$$\sum_{j=1}^{n} \frac{d(n-j)}{j} \, \mathrm{E}[\sum_{f} D_{j,f+1}] =$$

$$E[\sum_{f} \sum_{j=1}^{n} \frac{d(n-j)}{j} \, D_{j,f+1}]$$

6 Reforming the tax code

Here's another way to think of this last expression: every face f gets charged a "tax" of $\frac{d(n-j)}{j}$ for being destroyed at insertion $j+1$. The last expression is the total tax we'd expect to collect. Now what if we reform the tax code so that we tax each face $\frac{d(n-i)}{i}$ for being created at insertion i instead. Although the citezens probably won't notice the difference, the government will collect more money: we'll tax more individual faces, and, since $i \leq j$, each f gets charged at least as much ($\frac{d(n-i)}{i} \geq \frac{d(n-j)}{j}$ for $i \leq j$). Hence:

$$\mathrm{E}[X] \leq \mathrm{E}[\sum_{f} \sum_{i=1}^{n} \frac{d(n-i)}{i} \, C_{i,f}]$$

Which, re-exchanging E and $\sum$, means that

$$\mathrm{E}[X] \leq \sum_{i=i}^{n} \frac{d(n-i)}{i} \, \mathrm{E}[C_i]$$

7 Putting it all together

And we have a bound on $\mathrm{E}[C_i]$, thanks again to the standard backwards analysis argument. In R^3,

$$\mathrm{E}[C_i] = O(1)$$

Putting this together, we get, in R^3,

$$\mathrm{E}[X] \leq \sum_{i=i}^{n} \frac{3(n-i)}{i} \times O(1) =$$

$$O(n)\sum_{i=i}^{n} 1/i = O(n \lg n)$$

And in dimension d,

$$\mathrm{E}[C_i] \leq d/i \; O(i^{\lfloor d/2 \rfloor})$$

So that our bound comes out to be

$$\mathrm{E}[X] \leq \sum_{i=i}^{n} \frac{d(n-i)}{i} \times d/i \; O(i^{\lfloor d/2 \rfloor}) =$$

$$\sum_{i=i}^{n} d^2 \, \frac{n-i}{i^2} \, O(i^{\lfloor d/2 \rfloor}) =$$

$$O(d^2 n) \sum_{i=i}^{n} O(i^{\lfloor d/2 \rfloor - 2}) =$$

$$O(n^{\lfloor d/2 \rfloor})$$