1. Show that the Householder matrix $H_v = I - 2\frac{vv^T}{v^Tv}$ for $0 \neq v \in \mathbb{R}^n$ is *involutary*, meaning $H_v^2 = I$. What is the inverse of H_v?

2. Use the Householder transformation to compute the QR factorization of the matrix in Example 5.2. Do you obtain the same QR factorization as the Gram-Schmidt method?

3. Let $x, y \in \mathbb{R}^n$ with $x \neq y$ and $\|x\|_2 = \|y\|_2$, find a Householder transformation H_v such that $H_v x = y$. (*Hint: see pages 100-101.*)

4. Suppose we consider $a \in \mathbb{R}^n$ as an $n \times 1$ matrix. Write out its reduced QR factorization explicitly.

5. (a) Take $A \in \mathbb{R}^{m\times n}$ and suppose we apply the Cholesky factorization to obtain $A^TA = LL^T$. Define $Q = A(L^T)^{-1}$. Show that the columns of Q are orthonormal.

   (b) Based on (a), suggest a relationship between the Cholesky factorization of A^TA and the QR factorization of A.

6. Ranking sport teams. Suppose we have four college teams, call T1, T2, T3 and T4. These four teams play each other with the following outcomes:

   - T1 beats T2 by 4 points: 21 to 17.
   - T3 beats T1 by 9 points: 27 to 18.
   - T1 beats T4 by 6 points: 16 to 10.
   - T3 beats T4 by 3 points: 10 to 7.
   - T2 beats T4 by 7 points: 17 to 10.

   To determine ranking points r_1, r_2, r_3, r_4 for each team, we do a least squares fit to the overdetermined system:

$$\begin{aligned} r_1 - r_2 &= 4, \\ r_3 - r_1 &= 9, \\ r_1 - r_4 &= 6, \\ r_3 - r_4 &= 3, \\ r_2 - r_4 &= 7. \end{aligned}$$

   In addition, we fix the total number of ranking points, i.e., $r_1 + r_2 + r_3 + r_4 = 100$. Find the values of r_1, r_2, r_3, r_4 that most closely satisfy these equations, and based on your results rank the four teams.[1]

[1] This method of ranking sport teams is a simplification of one introduced by Ke Massey in 1997. It has evolved into a part of the famous BCS (Bowl Championship Series) model for ranking college football teams and is one factor in determining which teams play in bowl games.