Analyzing linear systems – §4.3

* vector and matrix norms (§4.3.1)

Def. a vector ~~form~~ norm is a function $\|\cdot\|: \mathbb{R}^n \to [0, \infty)$ satisfying that for any $x, y \in \mathbb{R}^n$

1) $\|x\| \geq 0$, and $\|x\| = 0$ if and only if $x = 0$
2) $\|\alpha x\| = |\alpha| \cdot \|x\|$, where α is a scalar
3) $\|x + y\| \leq \|x\| + \|y\|$

"Norm" is a metric to measure the "size"/"length" of a vector.

Commonly used vector norms: if $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

$$\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \quad \text{for } p \geq 0$$

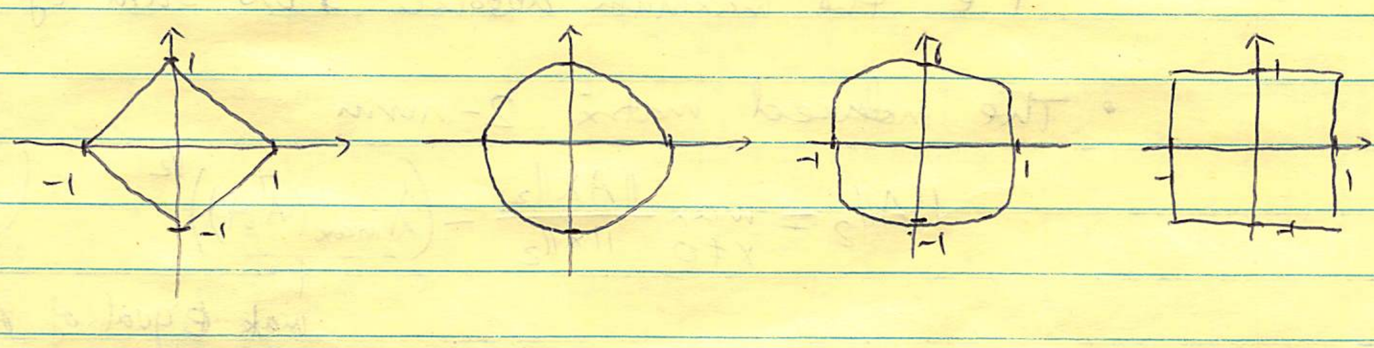
For examples: $p = 1$, $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

$p = 2$, $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2 \right)^{1/2} = \sqrt{x^T x}$

$p = \infty$, $\|x\|_\infty = \max_{1 \leq i \leq n} \{ |x_i| \}$

The set of $\{ x \in \mathbb{R}^n, \|x\|_p = 1 \}$ for different p — "geometry of norms"

$\|x\|_1 = 1$ $\quad$ $\|x\|_2 = 1$ $\quad$ $\|x\|_3 = 1$ $\quad$ $\|x\|_\infty = 1$



<u>Equivalency</u>: Two vector norms $\|\cdot\|_p$ and $\|\cdot\|_q$ are <u>equivalent</u> if there exists constants α and β such that

$$\alpha \|x\|_p \le \|x\|_q \le \beta \|x\|_p$$

for all $x \in \mathbb{R}^n$

<u>Thm</u> All norms on $\mathbb{R}^n$ are equivalent.

<u>Def</u>. An <u>induced matrix norm</u> (of $A \in \mathbb{R}^{m\times n}$) is a function $\mathbb{R}^{m\times n} \to [0,\infty)$ induced by a vector norm $\|\cdot\|$, given by

$$\|A\| = \max_{x\neq 0} \frac{\|Ax\|}{\|x\|}$$

Examples.

- The induced matrix 1-norm

$$\|A\|_1 = \max_{x\neq 0} \frac{\|Ax\|_1}{\|x\|_1} = \max_{1\le j\le n} \left\{ \sum_{i=1}^{m} |a_{ij}| \right\}$$

i.e, the maximum absolute column sum of A.

- The induced matrix ∞-norm

$$\|A\|_\infty = \max_{x\neq 0} \frac{\|Ax\|_\infty}{\|x\|_\infty} = \max_{1\le i\le m} \left\{ \sum_{j=1}^{n} |a_{ij}| \right\}$$

i.e. the maximum absolute row sum of A.

- The induced matrix 2-norm

$$\|A\|_2 = \max_{x\neq 0} \frac{\|Ax\|_2}{\|x\|_2} = \left(\underbrace{\lambda_{max}(A^TA)}_{\text{max eigval of } A^TA}\right)^{1/2}$$

(more in chap 7.)

* Sensitivity Analysis of linear systems — §4.3.2

Example #1 consider 2x2 linear system

$$\begin{pmatrix} 1 & 1.01 \\ 0.99 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 2.01 \\ 1.99 \end{pmatrix}$$

The solution is $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

Let us assume we make a perturbation of order 10^{-2} on the right-hand side to solve the linear system

$$\begin{pmatrix} 1. & 1.01 \\ 0.99 & 1 \end{pmatrix} \begin{pmatrix} \hat{x}_1 \\ \hat{x}_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$$

Then the solution becomes $\hat{x} = \begin{pmatrix} -200 \\ 200 \end{pmatrix}$.

It's amazing that we only made a small perturbation ("mistake"), but the solution has changed completed. — How to understand / analyze this pheaomanon? — without doubt, this is critically important!! #

To understand / interpret the example, let us consider ~~the~~ a perturbation of the original linear system $\underline{Ax = b}$ to

$$\underline{(A + \varepsilon E)\, x(\varepsilon)} = b + \varepsilon f \qquad (*)$$

where ε is a scalar to indicate the magnitude of the perturbation, E and f are constant matrix and vector, & $x(\varepsilon)$ is the sol. of the perturbed system $(*)$, a function of ε.

Differentiate the system (*) with respect to ε, we have

$$E\cdot x(\varepsilon) + (A+\varepsilon E)x'(\varepsilon) = f$$

At $\varepsilon = 0$, $\quad x'(0) = A^{-1}(f - E\,x(0))$.

Consider the Taylor expansion of $x(\varepsilon)$ at $\varepsilon = 0$:

$$x(\varepsilon) = \underbrace{x(0)}_{\text{the sol. of } Ax=b} + \varepsilon x'(0) + \underbrace{O(\varepsilon^2)}_{\text{higher-order terms of } \varepsilon}$$

Therefore, up to the first-order of ε:

$$x(\varepsilon) - x(0) = \varepsilon x'(0)$$
$$= \varepsilon A^{-1}(f - E\,x(0))$$

Taking the norm and using the triangular inequality

$$\|x(\varepsilon) - x(0)\| \le |\varepsilon| \cdot \|A^{-1}\| \left(\|f\| + \|E\,x(0)\|\right)$$
$$\le |\varepsilon| \, \|A^{-1}\| \left(\|f\| + \|E\| \cdot \|x(0)\|\right)$$

i.e. In term of the relative error:

$$\frac{\|x(\varepsilon) - x(0)\|}{\|x(0)\|} \le |\varepsilon| \, \|A^{-1}\| \cdot \left(\frac{\|f\|}{\|x(0)\|} + \|E\|\right)$$

$$\cancel{= |\varepsilon| \, \|A^{-1}\| \left(\frac{\|f\|}{\|A\|}\right.}$$

$$= |\varepsilon| \cdot \|A^{-1}\| \cdot \|A\| \cdot \left(\frac{\|f\|}{\|A\| \cdot \|x(0)\|} + \frac{\|E\|}{\|A\|}\right)$$

Note that $\quad Ax = Ax(0) = b$

$$\|b\| = \|A\,x(0)\| \le \|A\| \cdot \|x(0)\|$$

$$\frac{1}{\|x(0)\|} \le \frac{\|A\|}{\|b\|} \quad \text{or:} \quad \underline{\frac{1}{\|A\| \cdot \|x(0)\|} \le \frac{1}{\|b\|}}$$

Therefore, we have

$$\frac{\|X(\varepsilon)-X(0)\|}{\|X(0)\|} \le \varepsilon \, \underbrace{\|A\|\cdot\|A^{-1}\|}\left(\frac{\|f\|}{\|b\|}+\frac{\|E\|}{\|A\|}\right) \qquad (**)$$

By this upper bound on the relative error, we can say that if there is an $O(\varepsilon)$ changes (errors) to ~~the~~ A and/or b, then ~~there is~~ the ~~error~~ change/error could potentially be amplified by the factor $\|A\|\cdot\|A^{-1}\|$.
The quantity $\|A\|\cdot\|A^{-1}\|$ is called the condition number of the linear system.

Large condition number indicate that the solution to the system $Ax=b$ are unstable (sensitive) under the perturbation of A and/or b.

Example #1 (revisited)

We calculated that the condition number

$$\text{cond}(A)=\|A\|_\infty \|A^{-1}\|_\infty = 40401$$

$$\varepsilon = 10^{-2} = 0.01$$

$$\frac{\|x-\hat{x}\|_\infty}{\|x\|_\infty} = 201.00,$$

$$\varepsilon\cdot\text{Cond}(A)\cdot\frac{\|f\|_\infty}{\|b\|_\infty} = 202.00$$

The bound (**) indicate ~~the~~ such sensitivity to the perturbation in b. #

Geometrical interpolation of the condition number

$$\begin{aligned}
\mathrm{cond}(A) &= \|A\|\cdot\|A^{-1}\| \\
&= \left(\max_{x\neq 0}\frac{\|Ax\|}{\|x\|}\right)\cdot\left(\max_{x\neq 0}\frac{\|A^{-1}x\|}{\|x\|}\right) \\
&= \left(\max_{x\neq 0}\frac{\|Ax\|}{\|x\|}\right)\cdot\left(\max_{y\neq 0}\frac{\|y\|}{\|Ay\|}\right) \qquad \text{Let } y = A^{-1}x \\
&= \left(\max_{x\neq 0}\frac{\|Ax\|}{\|x\|}\right)\Big/\left(\min_{y\neq 0}\frac{\|Ay\|}{\|y\|}\right)
\end{aligned}$$

Therefore, the condition number of A measures the ratio of the largest and smallest distortion of any two points on the unit circle mapped under A.